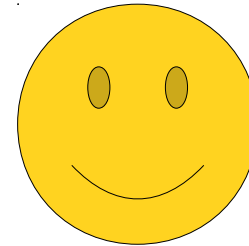
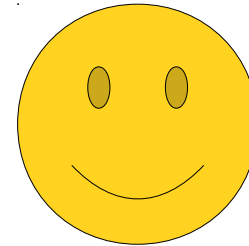


Guide to Negating Formulas

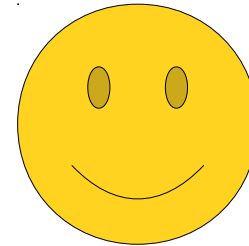
Hi everybody!



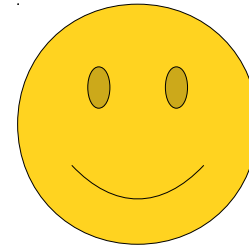
We spent a little bit of time in class talking about how to negate formulas in propositional or first-order logic.



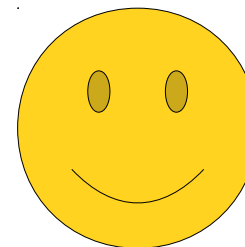
This is a really valuable skill! If you ever need to write a proof by contradiction or a proof by contrapositive, you'll need to know how to negate formulas.



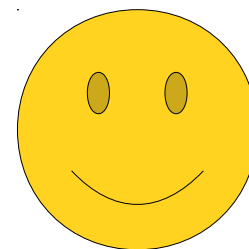
While this might seem a bit tricky at first,
the good news is that there's a nice,
mechanical way that you can negate
formulas!



There's still a bit of art to it, but by learning a few simple rules and how to apply them, you can negate just about anything!



For example...



$\neg p \wedge (q \vee r)$

Let's imagine that you want to negate this formula to the left.



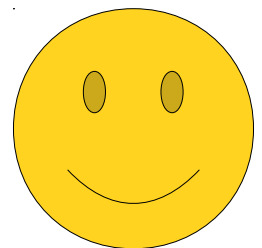
$(\neg p \wedge (q \vee r))$

To do so, we're going to begin by surrounding the formula in parentheses...



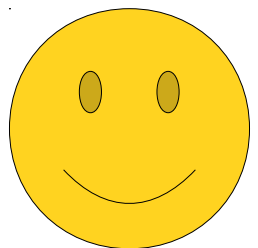
$$\neg(\neg p \wedge (q \vee r))$$

And putting a negation symbol
in front.



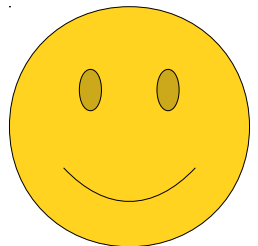
$$\neg(\neg p \wedge (q \vee r))$$

Technically speaking, this formula is the negation of the original formula, though it's hard to see exactly what this formula says.



$$\neg(\neg p \wedge (q \vee r))$$

Most of the time, when you need to find the negation of a formula, you're going to want to simplify it by pushing the negations inward.



$\neg(\neg p \wedge (q \vee r))$

The good news is that there are a number of rules we can use to do this.



$\neg(\neg p \wedge (q \vee r))$

If you remember from our lecture on propositional logic, we saw a series of rules for simplifying negations.

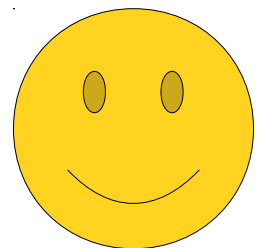


$$\neg(\neg p \wedge (q \vee r))$$

For example, we saw **de Morgan's laws**, which say that

$$\neg(A \wedge B) \equiv \neg A \vee \neg B$$

$$\neg(A \vee B) \equiv \neg A \wedge \neg B$$



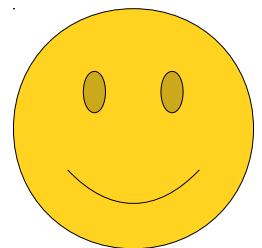
$\neg(A \vee B)$ $\neg(A \wedge B)$

 $\neg A \wedge \neg B$

 $\neg A \vee \neg B$

 $\neg(\neg p \wedge (q \vee r))$

I'm going to write this rules
up at the top of the screen.

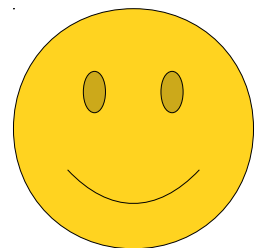


$\neg(A \vee B)$ $\neg(A \wedge B)$

 $\neg A \wedge \neg B$

 $\neg A \vee \neg B$ $\neg(\neg p \wedge (q \vee r))$

We haven't talked about this notation before, but the good news is that it's not too bad. Let's take a look at this.



$$\frac{\neg(A \vee B)}{\neg A \wedge \neg B}$$

$$\frac{\neg(A \wedge B)}{\neg A \vee \neg B}$$

Generally speaking, these rules look like this:

Before

After



$$\neg(\neg p \wedge (q \vee r))$$

$$\frac{\neg(A \vee B)}{\neg A \wedge \neg B}$$

$$\frac{\neg(A \wedge B)}{\neg A \vee \neg B}$$

A rule like this means "if you see *Before*, replace it with *After*."

Before

After



$$\neg(\neg p \wedge (q \vee r))$$

$$\frac{\neg(A \vee B)}{\neg A \wedge \neg B}$$

$$\frac{\neg(A \wedge B)}{\neg A \vee \neg B}$$

so let's look at this rule on
the right.



$$\neg(\neg p \wedge (q \vee r))$$

$$\frac{\neg(A \vee B)}{\neg A \wedge \neg B}$$

$$\frac{\neg(A \wedge B)}{\neg A \vee \neg B}$$

This rule says
"if you see something of the form
 $\neg(A \wedge B)$
replace it with
 $\neg A \vee \neg B$."

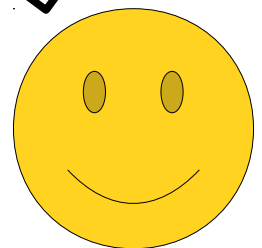


$$\neg(\neg p \wedge (q \vee r))$$

$$\frac{\neg(A \vee B)}{\neg A \wedge \neg B}$$
$$\frac{\neg(A \wedge B)}{\neg A \vee \neg B}$$

$$\neg(\neg p \wedge (q \vee r))$$

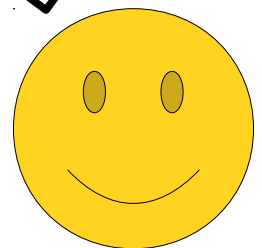
Take a look at our original formula. If you look at it closely, you'll see that it matches this particular pattern.



$$\frac{\neg(A \vee B)}{\neg A \wedge \neg B}$$
$$\frac{\neg(A \wedge B)}{\neg A \vee \neg B}$$

$$\neg(\neg p \wedge (q \vee r))$$

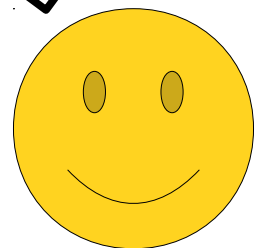
That means that we can apply
this rule and rewrite it...



$$\frac{\neg(A \vee B)}{\neg A \wedge \neg B}$$
$$\frac{\neg(A \wedge B)}{\neg A \vee \neg B}$$

$$\neg\neg p \vee \neg(q \vee r)$$

...so that it looks like this.

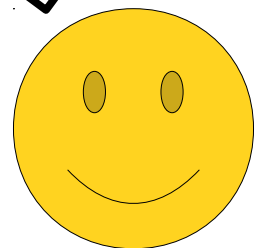


$$\frac{\neg(A \vee B)}{\neg A \wedge \neg B}$$

$$\frac{\neg(A \wedge B)}{\neg A \vee \neg B}$$

$$\neg\neg p \vee \neg(q \vee r)$$

Notice that we've "pushed" the negation deeper into the formula.



$\neg(A \vee B)$ $\neg(A \wedge B)$

 $\neg A \wedge \neg B$

 $\neg A \vee \neg B$ $\neg\neg p \vee \neg(q \vee r)$

Basically, we just need to keep applying these rule templates over and over until the formula is as simple as it can get.

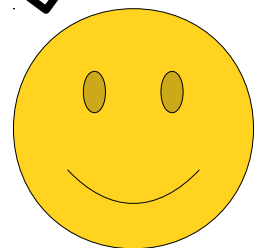


$$\frac{\neg(A \vee B)}{\neg A \wedge \neg B}$$

$$\frac{\neg(A \wedge B)}{\neg A \vee \neg B}$$

$$\neg\neg p \vee \neg(q \vee r)$$

so let's focus on this part of the formula for now.

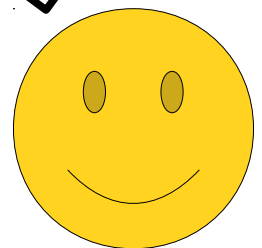


$$\frac{\neg(A \vee B)}{\neg A \wedge \neg B}$$

$$\frac{\neg(A \wedge B)}{\neg A \vee \neg B}$$

$$\neg\neg p \vee \neg(q \vee r)$$

Notice that it matches this particular template.

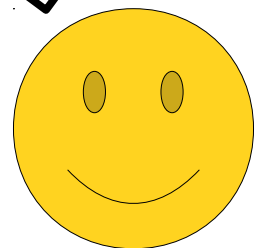


$$\frac{\neg(A \vee B)}{\neg A \wedge \neg B}$$

$$\frac{\neg(A \wedge B)}{\neg A \vee \neg B}$$

$$\neg\neg p \vee \neg(q \vee r)$$

so let's apply this rule to
simplify the formula!

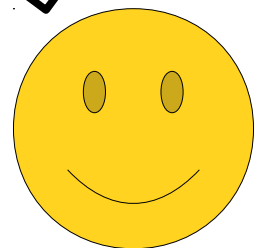


$$\frac{\neg(A \vee B)}{\neg A \wedge \neg B}$$

$$\frac{\neg(A \wedge B)}{\neg A \vee \neg B}$$

$$\neg\neg p \vee (\neg q \wedge \neg r)$$

Now we've got this, which has the negation pushed deeper into the formula! We're making progress!

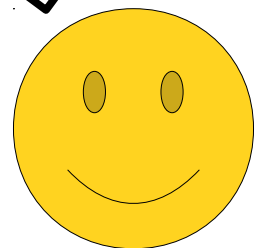


$$\frac{\neg(A \vee B)}{\neg A \wedge \neg B}$$

$$\frac{\neg(A \wedge B)}{\neg A \vee \neg B}$$

$$\neg\neg p \vee (\neg q \wedge \neg r)$$

So what about the other part of
this formula?



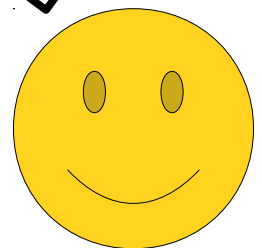
$\neg(A \vee B)$ $\neg(A \wedge B)$

 $\neg A \wedge \neg B$

 $\neg A \vee \neg B$

 $\neg\neg p \vee (\neg q \wedge \neg r)$

Well, this is a double-negation.
Intuitively, we'd expect these
two negations to cancel out.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg\neg p \vee (\neg q \wedge \neg r)$

Indeed, that's what happens.
We can write this down as the rule

 $\neg\neg A$

 A 

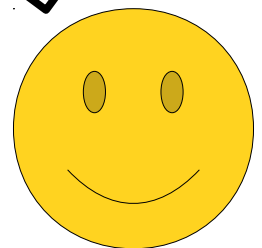
$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg\neg p \vee (\neg q \wedge \neg r)$

so let's go and apply this rule!



$\neg\neg A$

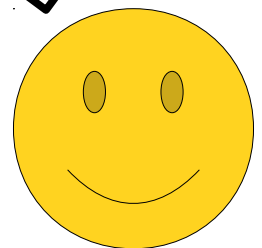
 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$

$p \vee (\neg q \wedge \neg r)$

Tada!



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $p \vee (\neg q \wedge \neg r)$

At this point we can't push the negations any deeper into the formula. They're directly applied to propositions, which can't be simplified.



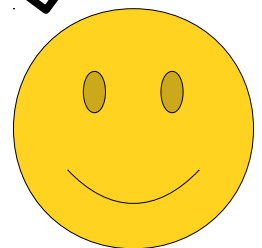
$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $p \vee (\neg q \wedge \neg r)$

so we're done!



$\neg\neg A$

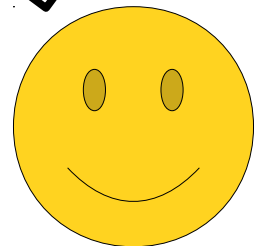
 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$

 $p \vee (\neg q \wedge \neg r)$

of course, before moving on,
we should be sure to check our
work!



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $p \vee (\neg q \wedge \neg r)$ $\neg p \wedge (q \vee r)$

Here's the original formula.
(Think of this as the "before" in
the "before and after" shots.)



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $p \vee (\neg q \wedge \neg r)$ $\neg p \wedge (q \vee r)$

Is there some way we can check whether we negated this formula properly?



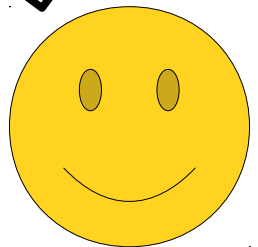
$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $p \vee (\neg q \wedge \neg r)$ $\neg p \wedge (q \vee r)$

Yes, actually! There are several.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $p \vee (\neg q \wedge \neg r)$ $\neg p \wedge (q \vee r)$

Let's begin with an intuitive check.
What do each of these formulas
actually mean?



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $p \vee (\neg q \wedge \neg r)$ $\neg p \wedge (q \vee r)$

This formula means "p is true,
or both q and r are false."



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $p \vee (\neg q \wedge \neg r)$ $\neg p \wedge (q \vee r)$

This formula means "p is false and at least one of q and r is true."



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $p \vee (\neg q \wedge \neg r)$ $\neg p \wedge (q \vee r)$

Intuitively, these statements seem to check out as opposites.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $p \vee (\neg q \wedge \neg r)$ $\neg p \wedge (q \vee r)$

If the first one is true, then either p is true (so $\neg p$ is false), or q and r are false (so $q \vee r$ is false.)



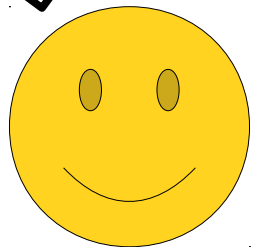
$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $p \vee (\neg q \wedge \neg r)$ $\neg p \wedge (q \vee r)$

similarly, if the second one is true, then $\neg p$ is true (so p is false), and at least one of q and r are true (so $\neg q \wedge \neg r$ is false).



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $p \vee (\neg q \wedge \neg r)$ $\neg p \wedge (q \vee r)$

so in that sense, intuitively,
everything checks out.



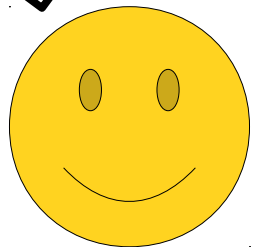
$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $p \vee (\neg q \wedge \neg r)$ $\neg p \wedge (q \vee r)$

Of course, a lot of things that make intuitive sense aren't right, so perhaps we should check this in a different way.



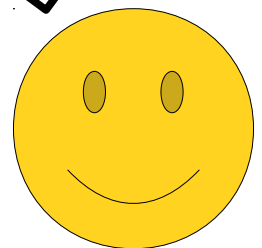
$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $p \vee (\neg q \wedge \neg r)$ $\neg p \wedge (q \vee r)$

When we're dealing with propositional logic, we can always check the truth tables for these formulas and see how they compare!



$\neg\neg A$

A

$p \vee (\neg q \wedge \neg r)$

p	q	r	$(p \vee (\neg q \wedge \neg r))$
F	F	F	T
F	F	T	F
F	T	F	F
F	T	T	F
T	F	F	T
T	F	T	T
T	T	F	T
T	T	T	T

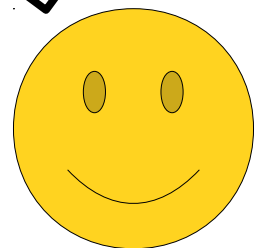
$\neg p \wedge (q \vee r)$

p	q	r	$(\neg p \wedge (q \vee r))$
F	F	F	F
F	F	T	T
F	T	F	T
F	T	T	T
T	F	F	F
T	F	T	F
T	T	F	F
T	T	T	F

$p \vee (\neg q \wedge \neg r)$

$\neg p \wedge (q \vee r)$

If you punch in these two formulas, here's the truth tables that you'll get back.



$\neg\neg A$

A

$p \vee (\neg q \wedge \neg r)$

p	q	r	$(p \vee (\neg q \wedge \neg r))$
F	F	F	T
F	F	T	F
F	T	F	F
F	T	T	F
T	F	F	T
T	F	T	T
T	T	F	T
T	T	T	T

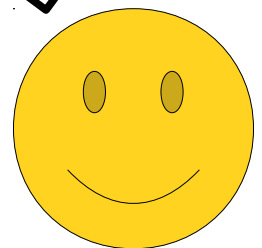
$\neg p \wedge (q \vee r)$

p	q	r	$(\neg p \wedge (q \vee r))$
F	F	F	F
F	F	T	T
F	T	F	T
F	T	T	T
T	F	F	F
T	F	T	F
T	T	F	F
T	T	T	F

$p \vee (\neg q \wedge \neg r)$

$\neg p \wedge (q \vee r)$

Notice that, going row by row, the truth values are opposites of one another. That means that they're negations of one another!



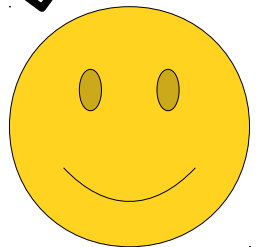
$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $p \vee (\neg q \wedge \neg r)$ $\neg p \wedge (q \vee r)$

There's actually a different way to use the truth table tool to check if two formulas are negations of one another.



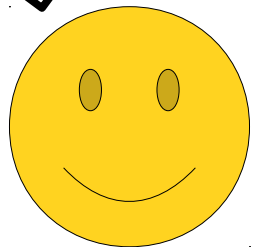
$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $p \vee (\neg q \wedge \neg r)$ $\neg p \wedge (q \vee r)$

Imagine that you have two formulas A and B . If A is a negation of B , then any time A is true, B is false and vice-versa.



$\neg\neg A$

 A $\neg(A \vee B)$

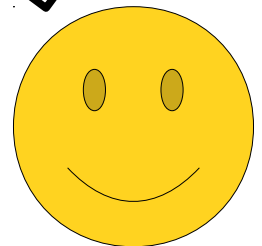
 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $p \vee (\neg q \wedge \neg r)$ $\neg p \wedge (q \vee r)$

That means that the formula

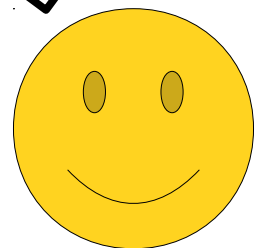
 $A \leftrightarrow B$

should always evaluate to false.



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $p \vee (\neg q \wedge \neg r)$ $\neg p \wedge (q \vee r)$

so in our case, we can try
punching in this formula into
the truth table tool.

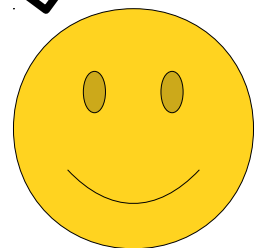
 $(p \vee (\neg q \wedge \neg r)) \leftrightarrow (\neg p \wedge (q \vee r))$ 

$\neg\neg A$ A $\neg(A \vee B)$ $\neg(A \wedge B)$ $\neg A$

p	q	r	$((p \vee (\neg q \wedge \neg r)) \leftrightarrow (\neg p \wedge (q \vee r)))$
F	F	F	F
F	F	T	F
F	T	F	F
F	T	T	F
T	F	F	F
T	F	T	F
T	T	F	F
T	T	T	F

 $p \vee (\neg q \wedge \neg r)$ $\neg p \wedge (q \vee r)$

If we do, here's what we get back.

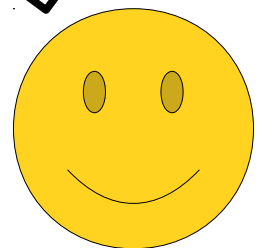
$$(p \vee (\neg q \wedge \neg r)) \leftrightarrow (\neg p \wedge (q \vee r))$$


$\neg\neg A$ A $\neg(A \vee B)$ $\neg(A \wedge B)$ $\neg A$

p	q	r	$((p \vee (\neg q \wedge \neg r)) \leftrightarrow (\neg p \wedge (q \vee r)))$
F	F	F	F
F	F	T	F
F	T	F	F
F	T	T	F
T	F	F	F
T	F	T	F
T	T	F	F
T	T	T	F

 $p \vee (\neg q \wedge \neg r)$ $\neg p \wedge (q \vee r)$

Since the formula is always false, we know that the two formulas must be negations of one another. Nifty!

$$(p \vee (\neg q \wedge \neg r)) \leftrightarrow (\neg p \wedge (q \vee r))$$


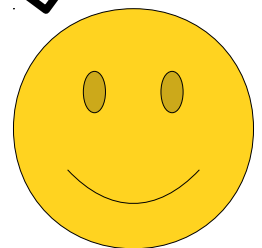
$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$

That wasn't so bad! Let's
do another one.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $(p \vee q) \wedge (\neg p \wedge \neg q)$

Let's try taking the negation of this formula.



$\neg\neg A$

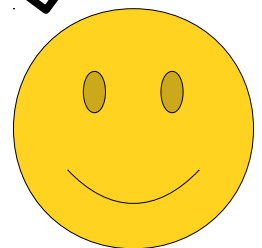
 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$

 $(p \vee q) \wedge (\neg p \wedge \neg q)$

Actually, before we do this together, why don't you try doing it yourself first!



$\neg\neg A$

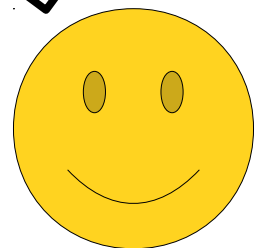
 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$

 $(p \vee q) \wedge (\neg p \wedge \neg q)$

Take a minute or two to walk through this one on your own. See what you come up with!



$\neg\neg A$

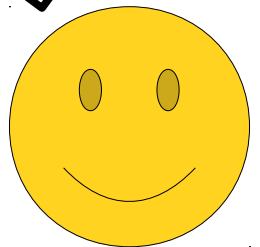
 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$

 $(p \vee q) \wedge (\neg p \wedge \neg q)$

So did you do it? If not, you really should. It's good practice!



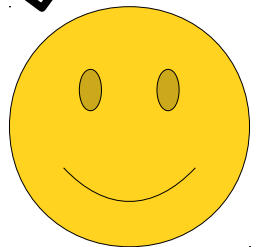
$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $(p \vee q) \wedge (\neg p \wedge \neg q)$

So you did it? You've got an answer? Great! Let's work through this one together.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $((p \vee q) \wedge (\neg p \wedge \neg q))$

As before, we begin by surrounding it in parentheses...



$\neg\neg A$

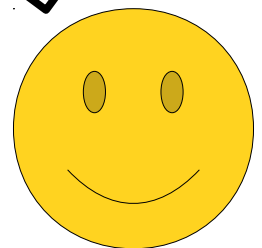
 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$

 $\neg((p \vee q) \wedge (\neg p \wedge \neg q))$

...and putting a negation at the front.



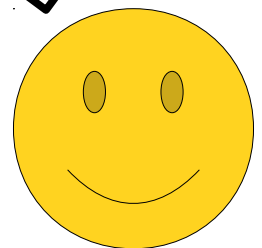
$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$
$$\neg((p \vee q) \wedge (\neg p \wedge \neg q))$$

Now, we're going to keep applying rules to push the negation deeper and deeper into the expression.



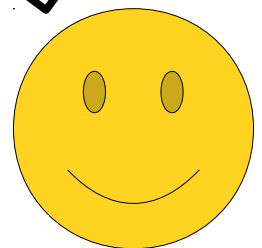
$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$
$$\neg((p \vee q) \wedge (\neg p \wedge \neg q))$$

Looking over the structure of this formula, we can see that it's a negation of two things ANDed together.



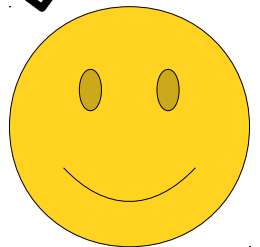
$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg((p \vee q) \wedge (\neg p \wedge \neg q))$

That means that we want to use this rule.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg((p \vee q) \wedge (\neg p \wedge \neg q))$

Color-coding things to make it easier to see which part is which, we can now apply the template...



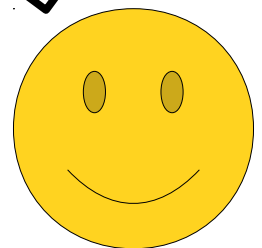
$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(p \vee q) \vee \neg(\neg p \wedge \neg q)$

...which gives us this initial simplification.



$\neg\neg A$

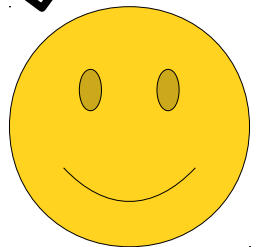
 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$

$$\neg(p \vee q) \vee \neg(\neg p \wedge \neg q)$$

Now, we can keep repeating this process on each smaller part of the expression.



$\neg\neg A$

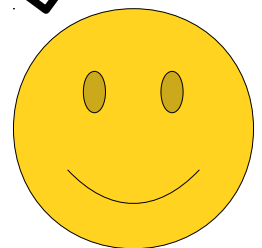
 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$

 $\neg(p \vee q) \vee \neg(\neg p \wedge \neg q)$

Just because we can, let's start with this part.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(p \vee q) \vee \neg(\neg p \wedge \neg q)$

That matches the same rule as before!



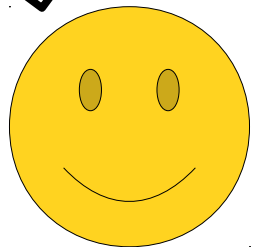
$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(p \vee q) \vee \neg(\neg p \wedge \neg q)$

So we can color-code things...



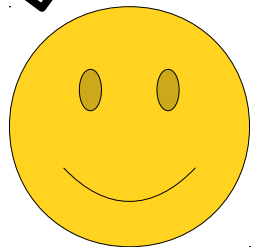
$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(p \vee q) \vee (\neg\neg p \vee \neg\neg q)$

...and apply the template.



$\neg\neg A$

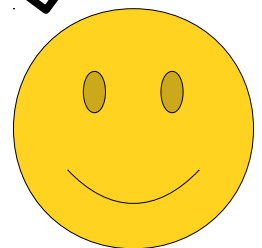
 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$

 $\neg(p \vee q) \vee (\neg\neg p \vee \neg\neg q)$

Progress!



$\neg\neg A$

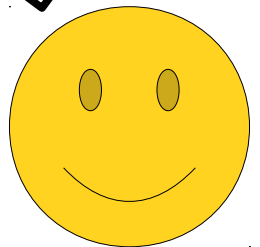
 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$

 $\neg(p \vee q) \vee (\neg\neg p \vee \neg\neg q)$

A reasonable next step would be to look at these two subexpressions, each of which is a double-negation.



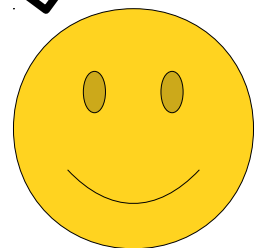
$$\frac{\neg\neg A}{A}$$

$$\frac{\neg(A \vee B)}{\neg A \wedge \neg B}$$

$$\frac{\neg(A \wedge B)}{\neg A \vee \neg B}$$

$$\neg(p \vee q) \vee (\neg\neg p \vee \neg\neg q)$$

Here's the template...



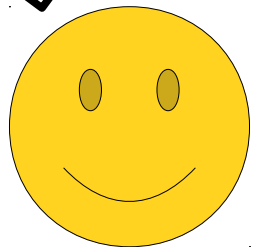
$$\frac{\neg\neg A}{A}$$

$$\frac{\neg(A \vee B)}{\neg A \wedge \neg B}$$

$$\frac{\neg(A \wedge B)}{\neg A \vee \neg B}$$

$$\neg(p \vee q) \vee (p \vee q)$$

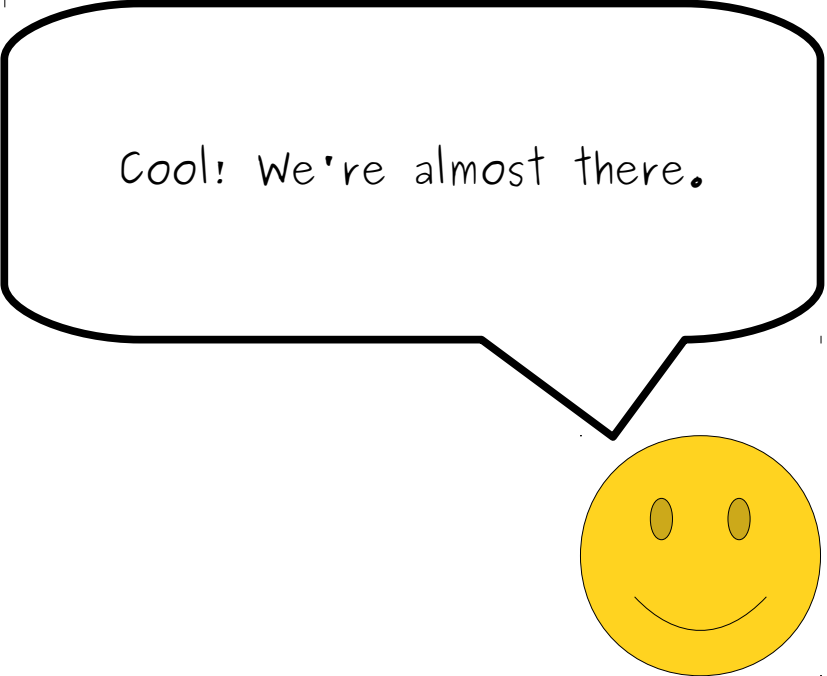
...and here's the result!



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(p \vee q) \vee (p \vee q)$ 

Cool! We're almost there.

$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(p \vee q) \vee (p \vee q)$

Our last step is to deal with
this part of the expression.



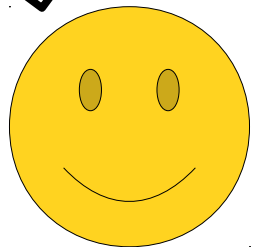
$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(p \vee q) \vee (p \vee q)$

That matches this template.



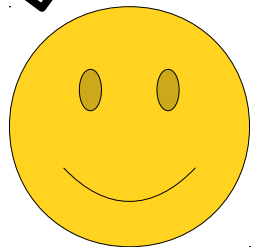
$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(p \vee q) \vee (p \vee q)$

So we see how it matches...



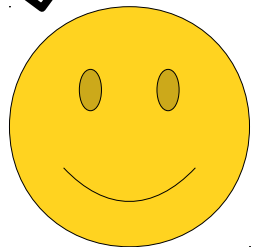
$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $(\neg p \wedge \neg q) \vee (p \vee q)$

...and apply the rule!



$\neg\neg A$

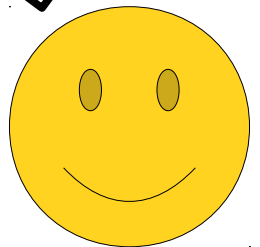
 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$

 $(\neg p \wedge \neg q) \vee (p \vee q)$

That's as deep as the negations
are going to go. We're done!



$\neg\neg A$

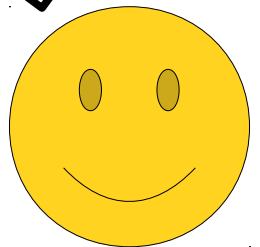
 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$

 $(\neg p \wedge \neg q) \vee (p \vee q)$

of course, we have to check
our work.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $(\neg p \wedge \neg q) \vee (p \vee q)$ $(p \vee q) \wedge (\neg p \wedge \neg q)$

Here's where we started.



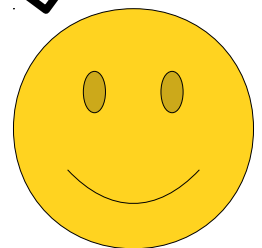
$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $(\neg p \wedge \neg q) \vee (p \vee q)$ $(p \vee q) \wedge (\neg p \wedge \neg q)$

Why don't you go use the truth table tool to check whether these statements actually are negations of one another?



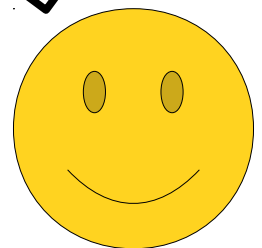
$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$

Okay! So let's recap where we are right now.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$

We have these nice and handy templates available for simplifying expressions with negations in them.



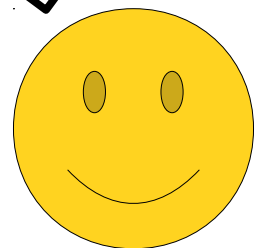
$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$

We've talked about three ways
that you can check your work.



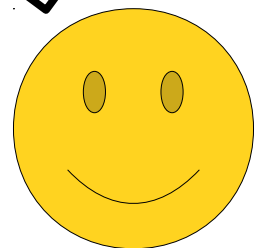
$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$

And we've gone through two examples!



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$

However, the rules above don't cover all possible formulas.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $p \wedge q \rightarrow r$

For example, how would we negate this formula?



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $p \wedge q \rightarrow r$

First, let's refresh how to negate an implication.



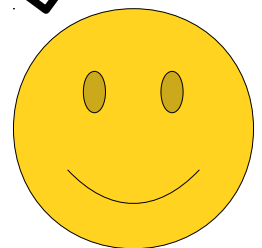
$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $p \wedge q \rightarrow r$

Actually, before we review that one directly, why don't you review your notes and see how you negate an implication?



$\neg\neg A$

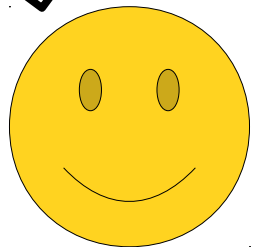
 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $p \wedge q \rightarrow r$

What goes below this line?

 $\neg(A \rightarrow B)$

 $??$ 

$\neg\neg A$

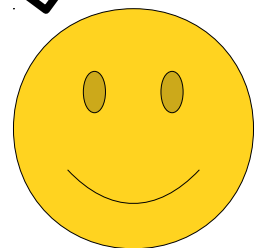
 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $p \wedge q \rightarrow r$

Here's the answer:

 $\neg(A \rightarrow B)$

 $A \wedge \neg B$ 

$\neg\neg A$

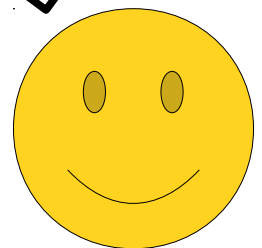
 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $p \wedge q \rightarrow r$

Remember, the only way for an implication to be false is if the antecedent (A) is true and the consequent (B) is false!



$\neg\neg A$

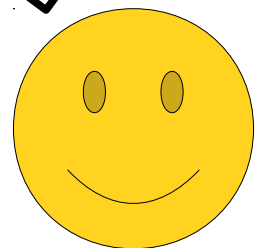
 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $p \wedge q \rightarrow r$

One of the most common mistakes we see people make when negating formulas is to get this wrong, so commit this one to memory!



$\neg\neg A$

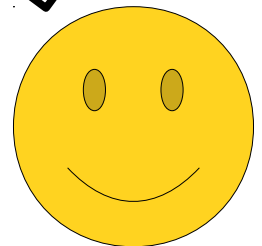
 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $p \wedge q \rightarrow r$

Okay! Now that we have our rule, let's see how to negate this formula.



$\neg\neg A$

 A $\neg(A \vee B)$

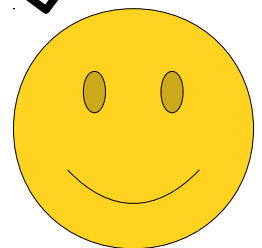
 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$

 $(p \wedge q \rightarrow r)$

As before, we parenthesize it...



$\neg\neg A$

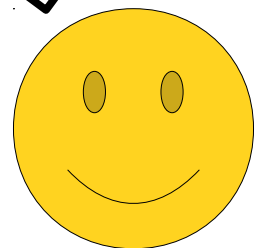
 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(p \wedge q \rightarrow r)$

...and put a negation out front.



$\neg\neg A$

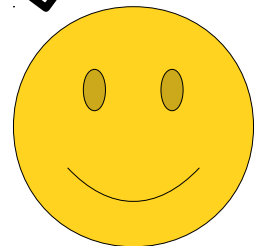
 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(p \wedge q \rightarrow r)$

Now, we need to see what template to apply.



$\neg\neg A$

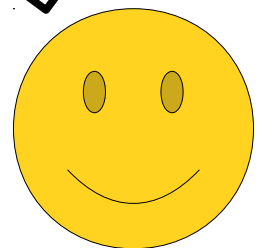
 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(p \wedge q \rightarrow r)$

Here, we're going to run into an operator precedence issue, which we haven't seen yet.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(p \wedge q \rightarrow r)$ $\neg((p \wedge q) \rightarrow r) \quad (?)$

Do we interpret the formula
like this?



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(p \wedge q \rightarrow r)$ $\neg((p \wedge q) \rightarrow r)$ (?) $\neg(p \wedge (q \rightarrow r))$ (?)

Or like this?



$\neg\neg A$

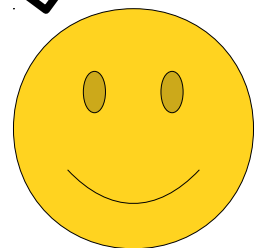
 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(p \wedge q \rightarrow r)$ $\neg((p \wedge q) \rightarrow r)$ (?) $\neg(p \wedge (q \rightarrow r))$ (?)

Before we tell you, why don't you look over your notes and see what you find?



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(p \wedge q \rightarrow r)$ $\neg((p \wedge q) \rightarrow r)$ (?) $\neg(p \wedge (q \rightarrow r))$ (?)

Okay... you've got a guess?



$\neg\neg A$

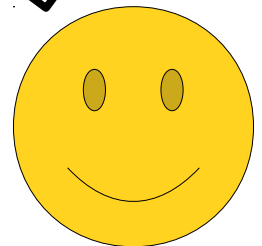
 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(p \wedge q \rightarrow r)$ $\neg((p \wedge q) \rightarrow r)$ ~~$\neg(p \wedge (q \rightarrow r))$~~ (?)

The top one has the right operator precedence.



$\neg\neg A$

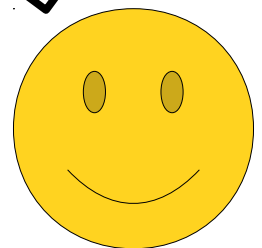
 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg((p \wedge q) \rightarrow r)$ $\neg((p \wedge q) \rightarrow r)$ ~~$\neg(p \wedge (q \rightarrow r))$~~ (?)

Just to make it easier for us to remember that, let's go and add those parentheses into our top-level formula.



$\neg\neg A$

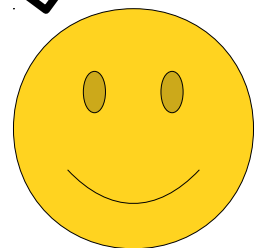
 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg((p \wedge q) \rightarrow r)$

So now we can look at our templates and see which one applies.



$\neg\neg A$

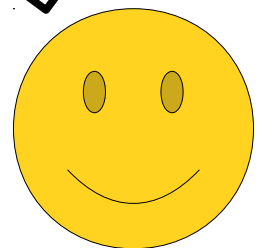
 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg((p \wedge q) \rightarrow r)$

Here, we're negating an implication, so this rule applies.



$\neg\neg A$

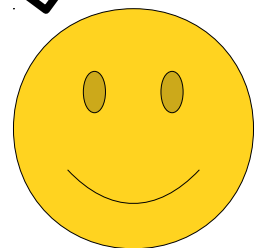
 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg((p \wedge q) \rightarrow r)$

so we find the correspondence...



$\neg\neg A$

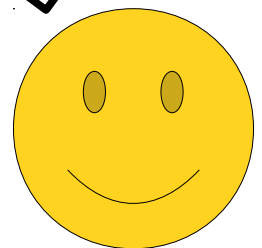
 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $(p \wedge q) \wedge \neg r$

...and apply the template!



$\neg\neg A$

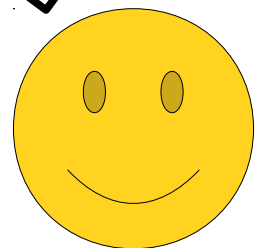
 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $(p \wedge q) \wedge \neg r$

That gives us this expression.



$\neg\neg A$

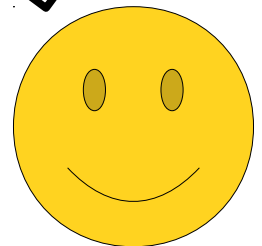
 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $p \wedge q \wedge \neg r$

We can actually drop the parentheses here, since everything is getting ANDed together and AND is associative.



$\neg\neg A$

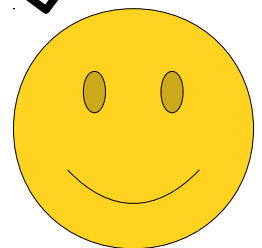
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 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $p \wedge q \wedge \neg r$

At this point, we can't simplify things any more, so we're done!



$\neg\neg A$

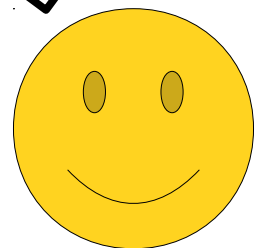
 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $p \wedge q \wedge \neg r$

As before, we should check our work.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $p \wedge q \wedge \neg r$ $p \wedge q \rightarrow r$

Here's the original formula.



$\neg\neg A$

 A $\neg(A \vee B)$

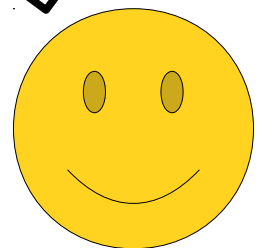
 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$

$$(p \wedge q \wedge \neg r) \\ \leftrightarrow \\ (p \wedge q \rightarrow r)$$

If we plug this expression into the truth table tool, we can see that these formulas are indeed negations of one another. (Try it!)



$\neg\neg A$

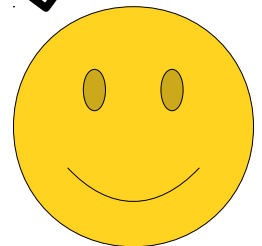
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 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $p \wedge q \wedge \neg r$ $p \wedge q \rightarrow r$

But, intuitively, why is that?



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $p \wedge q \wedge \neg r$ $p \wedge q \rightarrow r$

Notice that the original formula
(the one on the bottom) says
"p and q implies r."



$\neg\neg A$

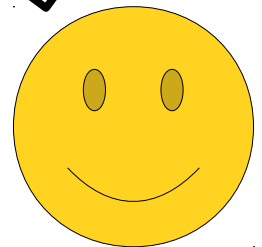
 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $p \wedge q \wedge \neg r$ $p \wedge q \rightarrow r$

That statement is going to be true unless the antecedent (p and q) is true and the consequent (r) is false.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $p \wedge q \wedge \neg r$ $p \wedge q \rightarrow r$

So take a look at what our negation (the top formula) says: it says "p and q are true, but r is false." It checks out!



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $p \wedge q \wedge \neg r$ $p \wedge q \rightarrow r$

Before we do another example, one detail that's worth noting is that we didn't even touch the antecedent.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $p \wedge q \wedge \neg r$ $p \wedge q \rightarrow r$

Notice that it's the same in both the original formula and the negation.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $p \wedge q \wedge \neg r$ $p \wedge q \rightarrow r$

When you're negating statements, you don't necessarily negate every part of them.



$\neg\neg A$

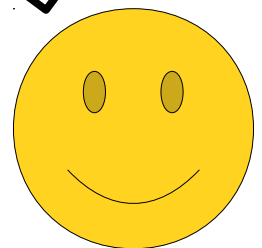
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 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $p \wedge q \wedge \neg r$ $p \wedge q \rightarrow r$

This is why these rules matter:
they show you what parts get
negated and what parts stay
the same.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $p \wedge q \wedge \neg r$ $p \wedge q \rightarrow r$

It's important to keep that in mind - one of the more common mistakes we see people make is negating way more than they should.



$\neg\neg A$

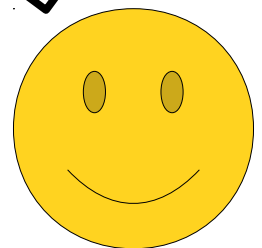
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 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $p \wedge q \wedge \neg r$ $p \wedge q \rightarrow r$

So, when in doubt, just keep applying the templates!



$\neg\neg A$

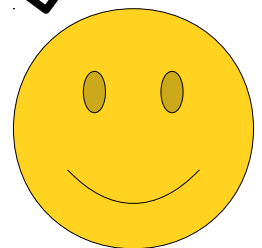
 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$

With that said, let's do another example.



$\neg\neg A$

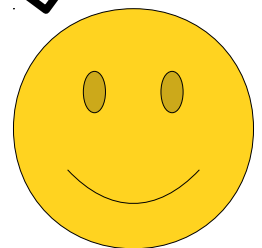
 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $p \rightarrow q \rightarrow r$

How might we negate this statement?



$\neg\neg A$

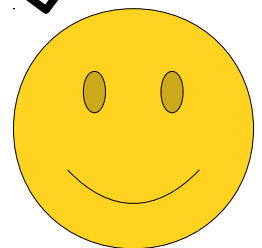
 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $(p \rightarrow q \rightarrow r)$

You know the drill! first we
parenthesize...



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(p \rightarrow q \rightarrow r)$

...then we stick a negation
out front.



$\neg\neg A$

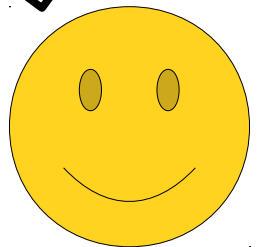
 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(p \rightarrow q \rightarrow r)$

Now we just need to apply templates.



$\neg\neg A$

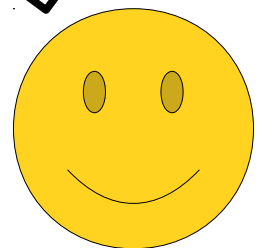
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 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(p \rightarrow q \rightarrow r)$

It should be clear that we need to use this template for implications, but how do we apply it?



$\neg\neg A$

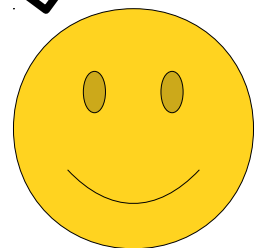
 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(p \rightarrow q \rightarrow r)$ $\neg((p \rightarrow q) \rightarrow r)$ (?) $\neg(p \rightarrow (q \rightarrow r))$ (?)

Specifically, which of these interpretations of the expression are correct?



$\neg\neg A$

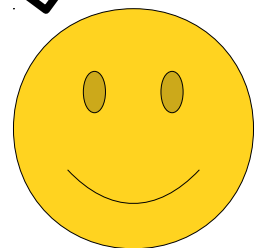
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 $A \wedge \neg B$ $\neg(p \rightarrow q \rightarrow r)$ $\neg((p \rightarrow q) \rightarrow r)$ (?) $\neg(p \rightarrow (q \rightarrow r))$ (?)

Take a minute to review your notes and make a guess. Seriously, try it! It's a good exercise.



$\neg\neg A$

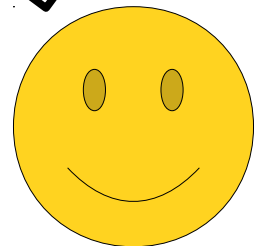
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 $A \wedge \neg B$ $\neg(p \rightarrow q \rightarrow r)$ $\neg((p \rightarrow q) \rightarrow r)$ (?) $\neg(p \rightarrow (q \rightarrow r))$ (?)

So you have a guess?
If not, go make one before
going on.



$\neg\neg A$

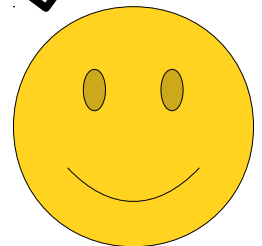
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 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(p \rightarrow q \rightarrow r)$ $\neg((p \rightarrow q) \rightarrow r)$ (?) $\neg(p \rightarrow (q \rightarrow r))$

so this is the correct interpretation. The \rightarrow operator is right-associative.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(p \rightarrow (q \rightarrow r))$ ~~$\neg((p \rightarrow q) \rightarrow r)$~~ (?) $\neg(p \rightarrow (q \rightarrow r))$

Let's go put those parentheses
in up here.



$\neg\neg A$

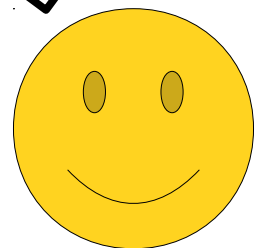
 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(p \rightarrow (q \rightarrow r))$

Okay! Now let's start applying templates.



$\neg\neg A$

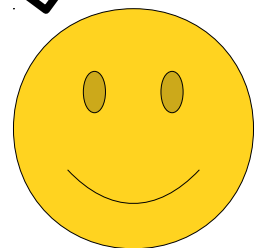
 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(p \rightarrow (q \rightarrow r))$

Color-coding the antecedent and consequent lets us see the structure...



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $p \wedge \neg(q \rightarrow r)$

...and makes it clear how we push the negation inward.



$\neg\neg A$

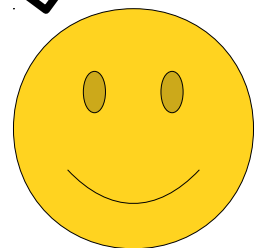
 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $p \wedge \neg(q \rightarrow r)$

Progress!



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $p \wedge \neg(q \rightarrow r)$

Now, we can apply this template another time.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $p \wedge \neg(q \rightarrow r)$

Before...



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $p \wedge q \wedge \neg r$

...and after!



$\neg\neg A$

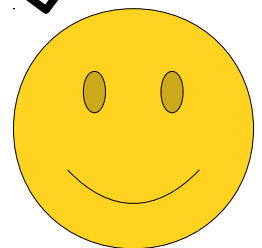
 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $p \wedge q \wedge \neg r$

so we're done!



$\neg\neg A$

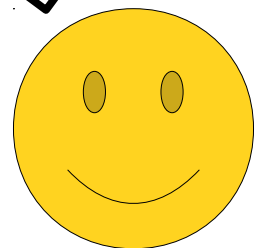
 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $p \wedge q \wedge \neg r$ $p \rightarrow q \rightarrow r$

Here's the original formula and the ultimate negation. You can (and should!) go check whether this is right using the truth table tool.



$\neg\neg A$

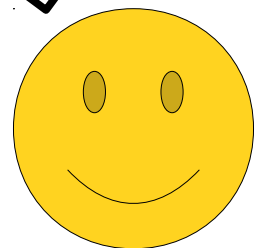
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 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $p \wedge q \wedge \neg r$ $p \rightarrow q \rightarrow r$

But hold on a second...
does this negation seem familiar?



$\neg\neg A$

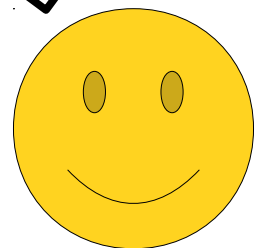
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 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $p \wedge q \wedge \neg r$ $p \rightarrow q \rightarrow r$

Hopefully it does - we saw earlier that it's the negation of

 $p \wedge q \rightarrow r!$ 

$\neg\neg A$

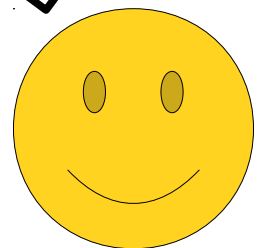
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 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $p \wedge q \wedge \neg r$ $p \rightarrow q \rightarrow r$

This is exciting, because it means we just discovered something new!



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $p \wedge q \wedge \neg r$ $p \rightarrow q \rightarrow r$
$$p \rightarrow q \rightarrow r \equiv p \wedge q \rightarrow r$$

Since these two formulas have the same negation, they must be equivalent to one another!



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $p \wedge q \wedge \neg r$ $p \rightarrow q \rightarrow r$
$$p \rightarrow q \rightarrow r \equiv p \wedge q \rightarrow r$$

This particular identity shows up in a lot of places, actually. Curious? Come ask us about it!



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$

Okay, so now we have a rule for negating implications. Great!



$\neg\neg A$

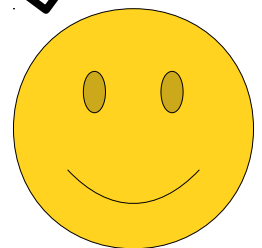
 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$

Before we move on, there's something important I'd like to address.



$\neg\neg A$

 A $\neg(A \vee B)$

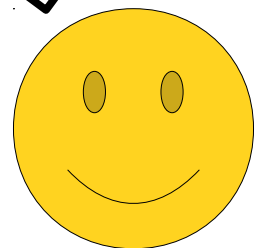
 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$

If you'll remember, in class we saw this equivalence:

$$\neg(A \wedge B) \equiv A \rightarrow \neg B$$



$\neg\neg A$

 A $\neg(A \vee B)$

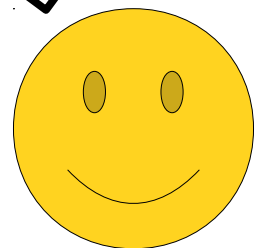
 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$

We can treat this as a new template for negating AND:

 $\neg(A \wedge B)$

 $A \rightarrow \neg B$ 

$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg(A \rightarrow B)$

Let's add it to the list.



$\neg\neg A$

 A $\neg(A \vee B)$

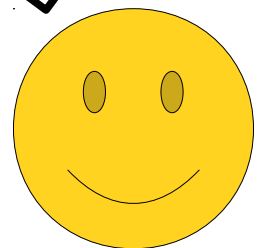
 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$

We now have two different rules
for negating ANDs.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

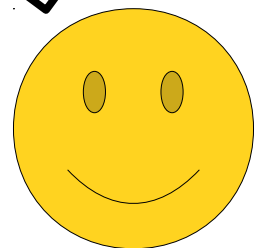
 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$

$$\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$$

$$\neg(A \wedge B) \leftrightarrow (A \rightarrow \neg B)$$

We can use the truth table tool to check that both of these rules are correct. Just punch in the formulas to the left and see what you get!



$\neg\neg A$

 A $\neg(A \vee B)$

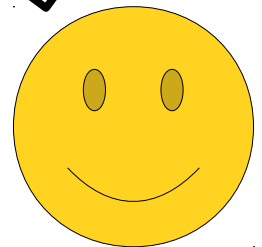
 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$

But what does it mean to have two different rules lying around for negating AND?



$\neg\neg A$

 A $\neg(A \vee B)$

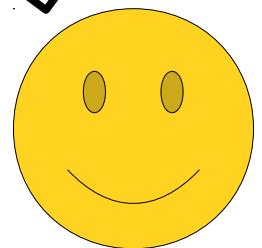
 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$

This means that if we want to negate an AND, we have two different ways of doing it.



$\neg\neg A$

 A $\neg(A \vee B)$

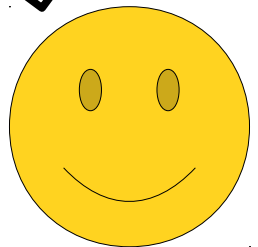
 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$

This template uses de Morgan's laws. It tends to be more useful when working in propositional logic.



$\neg\neg A$

 A $\neg(A \vee B)$

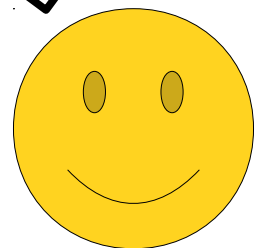
 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$

This template is based on the negation rule for implications. It's more useful in first-order logic.



$\neg\neg A$

 A $\neg(A \vee B)$

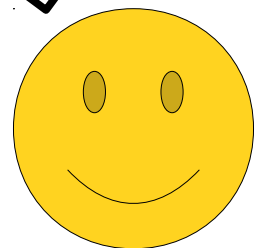
 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$

Now that we have several different negation rules available, we can talk about a key property of negations we've skipped up to this point.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$

$p \wedge q \wedge r$

Let's look at how to negate this formula.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$

$p \wedge q \wedge r$

Before we go over this one together, take a few minutes to negate this one on your own.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

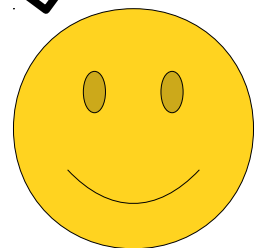
 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$

$p \wedge q \wedge r$

Really, it's a good exercise.
Don't continue onward until
you've done that.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$

$p \wedge q \wedge r$

So you negated it? Good!
Great! Let's do it together.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg(p \wedge q \wedge r)$

The first few steps should be pretty routine by this point – parenthesize and negate!



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(p \wedge q \wedge r)$

Now, we have a lot of different options here. We have two different templates for negating AND...



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(p \wedge q \wedge r)$ $\neg((p \wedge q) \wedge r)$ $\neg(p \wedge (q \wedge r))$

...and at the same time, there are two equally correct ways of parenthesizing this expression.



$\neg\neg A$

 A $\neg(A \vee B)$

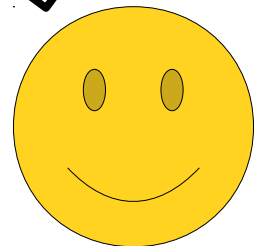
 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(p \wedge q \wedge r)$ $\neg((p \wedge q) \wedge r)$ $\neg(p \wedge (q \wedge r))$

Depending on which templates you use, and how you group things, you may end up with totally different answers!



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(p \wedge q \wedge r)$ $\neg((p \wedge q) \wedge r)$ $\neg(p \wedge (q \wedge r))$

Let's see what they look like.



$\neg\neg A$

 A $\neg(A \vee B)$

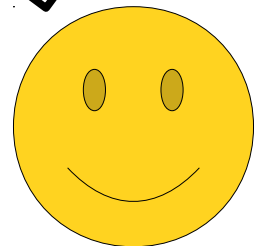
 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg(p \wedge (q \wedge r))$

For starters, let's parenthesize things like this.



$\neg\neg A$

 A $\neg(A \vee B)$

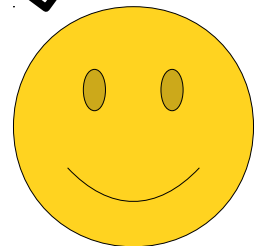
 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg(p \wedge (q \wedge r))$

Let's use this first template.
We can be old-school.



$\neg\neg A$

 A $\neg(A \vee B)$

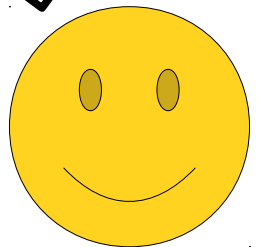
 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg(p \wedge (q \wedge r))$

Here's the "before" picture...



$\neg\neg A$

 A $\neg(A \vee B)$

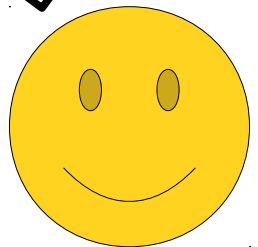
 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg p \vee \neg(q \wedge r)$

... and the "after" picture.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$

 $\neg p \vee \neg(q \wedge r)$

So now we have to simplify
the second part of this
expression.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg p \vee \neg(q \wedge r)$

Let's use this template.



$\neg\neg A$

 A $\neg(A \vee B)$

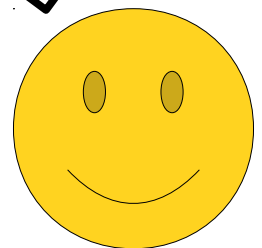
 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg p \vee \neg(q \wedge r)$

Before...



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg p \vee (q \rightarrow \neg r)$

... and after!



$\neg\neg A$

 A $\neg(A \vee B)$

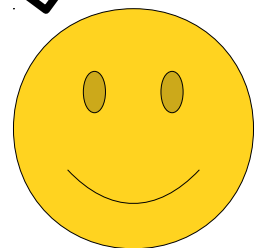
 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg p \vee (q \rightarrow \neg r)$

so this is one possible formula we could arrive at as the negation of the original formula.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg p \vee (q \rightarrow \neg r)$

This isn't all that easy to read.
Let's try negating it a different
way.



$\neg\neg A$

 A $\neg(A \vee B)$

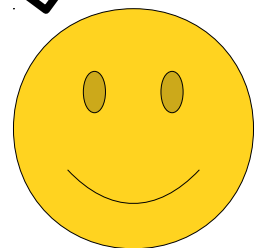
 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg p \vee (q \rightarrow \neg r)$
 $\neg(p \wedge q \wedge r)$

Resetting back to where we began...



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg p \vee (q \rightarrow \neg r)$
 $\neg((p \wedge q) \wedge r)$

Let's parenthesize it like this.



$\neg\neg A$

 A $\neg(A \vee B)$

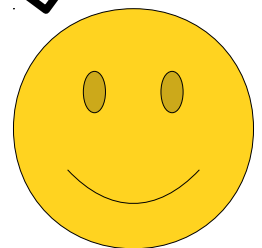
 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg p \vee (q \rightarrow \neg r)$
 $\neg((p \wedge q) \wedge r)$

Now, let's use this template.



$\neg\neg A$

 A $\neg(A \vee B)$

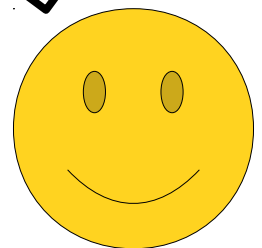
 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg p \vee (q \rightarrow \neg r)$
 $\neg((p \wedge q) \wedge r)$

That groups things like this...



$\neg\neg A$

 A $\neg(A \vee B)$

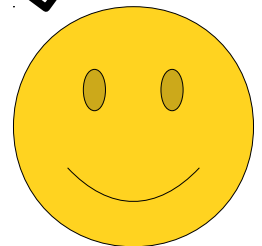
 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg p \vee (q \rightarrow \neg r)$
 $p \wedge q \rightarrow \neg r$

...giving us this result!



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$
$$\neg p \vee (q \rightarrow \neg r)$$
$$p \wedge q \rightarrow \neg r$$

And we're done! That was pretty fast. And we now have a very different formula than the first one!



$\neg\neg A$

 A $\neg(A \vee B)$

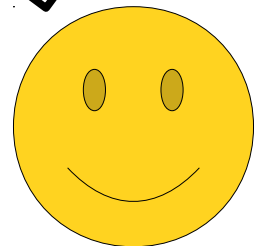
 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$
$$\neg p \vee (q \rightarrow \neg r)$$
$$p \wedge q \rightarrow \neg r$$

of course, this isn't the only way you could do this.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

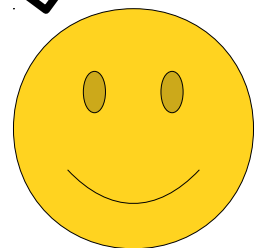
 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$

$\neg p \vee (q \rightarrow \neg r)$
 $p \wedge q \rightarrow \neg r$
 $\neg(p \wedge q \wedge r)$

Let's see another one.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

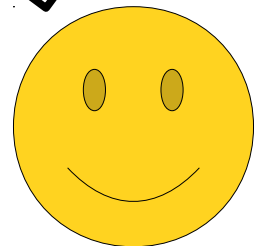
 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$

$\neg p \vee (q \rightarrow \neg r)$
 $p \wedge q \rightarrow \neg r$
 $\neg(p \wedge (q \wedge r))$

We'll group things this way.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

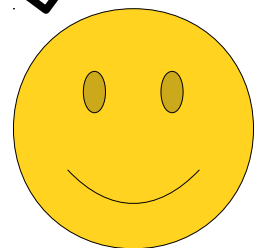
 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$

$\neg p \vee (q \rightarrow \neg r)$
 $p \wedge q \rightarrow \neg r$
 $\neg(p \wedge (q \wedge r))$

Now, we'll apply this template.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

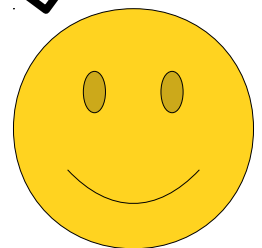
 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$

$\neg p \vee (q \rightarrow \neg r)$
 $p \wedge q \rightarrow \neg r$
 $\neg(p \wedge (q \wedge r))$

This groups things in a pretty different way from before...



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

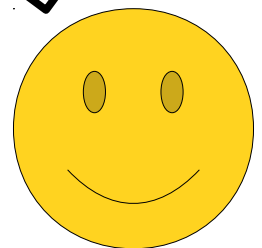
 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$

$\neg p \vee (q \rightarrow \neg r)$
 $p \wedge q \rightarrow \neg r$
 $p \rightarrow \neg(q \wedge r)$

...so we end up with something pretty different than before!



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$

$\neg p \vee (q \rightarrow \neg r)$
 $p \wedge q \rightarrow \neg r$
 $p \rightarrow \neg(q \wedge r)$

We still need to simplify that second part.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

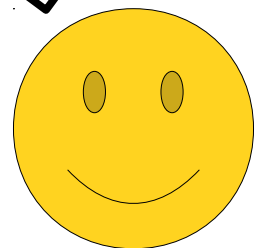
 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$

$\neg p \vee (q \rightarrow \neg r)$
 $p \wedge q \rightarrow \neg r$
 $p \rightarrow \neg(q \wedge r)$

Let's apply this template
again.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

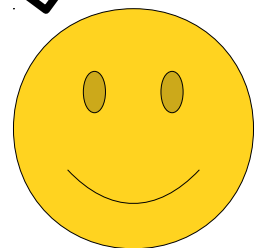
 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$

$\neg p \vee (q \rightarrow \neg r)$
 $p \wedge q \rightarrow \neg r$
 $p \rightarrow \neg(q \wedge r)$

That takes us from here...



$\neg\neg A$

 A $\neg(A \vee B)$

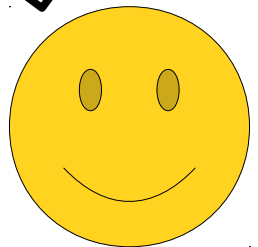
 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg p \vee (q \rightarrow \neg r)$ $p \wedge q \rightarrow \neg r$ $p \rightarrow q \rightarrow \neg r$

...to here.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$

$\neg p \vee (q \rightarrow \neg r)$
 $p \wedge q \rightarrow \neg r$
 $p \rightarrow q \rightarrow \neg r$

And we're done! Another
totally valid negation.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

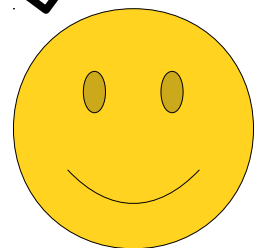
 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$

$\neg p \vee (q \rightarrow \neg r)$
 $p \wedge q \rightarrow \neg r$
 $p \rightarrow q \rightarrow \neg r$

Let's do one more.



$\neg\neg A$

 A $\neg(A \vee B)$

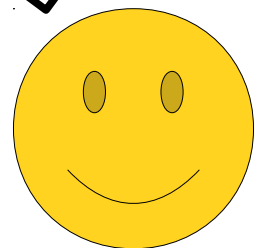
 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg p \vee (q \rightarrow \neg r)$ $p \wedge q \rightarrow \neg r$ $p \rightarrow q \rightarrow \neg r$ $\neg(p \wedge q \wedge r)$

We'll start here...



$\neg\neg A$

 A $\neg(A \vee B)$

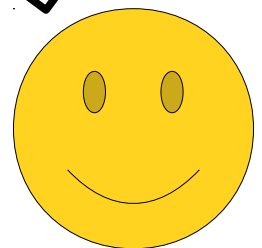
 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg p \vee (q \rightarrow \neg r)$ $p \wedge q \rightarrow \neg r$ $p \rightarrow q \rightarrow \neg r$ $\neg((p \wedge q) \wedge r)$

... and group things like this.



$\neg\neg A$

 A $\neg(A \vee B)$

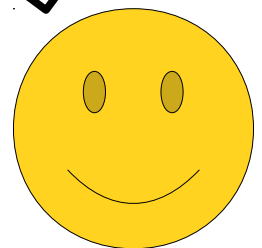
 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg p \vee (q \rightarrow \neg r)$ $p \wedge q \rightarrow \neg r$ $p \rightarrow q \rightarrow \neg r$ $\neg((p \wedge q) \wedge r)$

Let's use the first template,
for a change.



$\neg\neg A$

 A $\neg(A \vee B)$

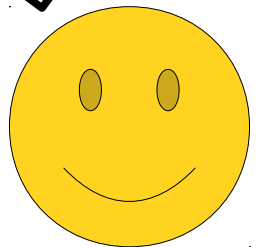
 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg p \vee (q \rightarrow \neg r)$ $p \wedge q \rightarrow \neg r$ $p \rightarrow q \rightarrow \neg r$ $\neg((p \wedge q) \wedge r)$

Here's where we start...



$\neg\neg A$

 A $\neg(A \vee B)$

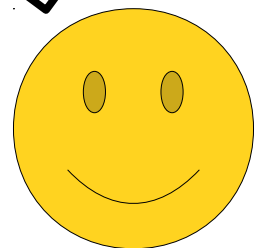
 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg p \vee (q \rightarrow \neg r)$ $p \wedge q \rightarrow \neg r$ $p \rightarrow q \rightarrow \neg r$ $\neg(p \wedge q) \vee \neg r$

...and where we end.



$\neg\neg A$

 A $\neg(A \vee B)$

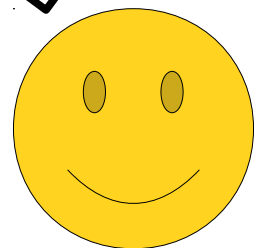
 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg p \vee (q \rightarrow \neg r)$ $p \wedge q \rightarrow \neg r$ $p \rightarrow q \rightarrow \neg r$ $\neg(p \wedge q) \vee \neg r$

Now, we need to fix up that first part.



$\neg\neg A$

 A $\neg(A \vee B)$

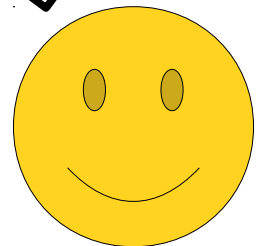
 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg p \vee (q \rightarrow \neg r)$ $p \wedge q \rightarrow \neg r$ $p \rightarrow q \rightarrow \neg r$ $\neg(p \wedge q) \vee \neg r$

We'll use this template
again...



$\neg\neg A$

 A $\neg(A \vee B)$

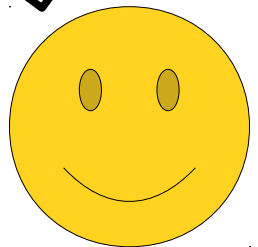
 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg p \vee (q \rightarrow \neg r)$ $p \wedge q \rightarrow \neg r$ $p \rightarrow q \rightarrow \neg r$ $\neg(p \wedge q) \vee \neg r$

...taking us from here...



$\neg\neg A$

 A $\neg(A \vee B)$

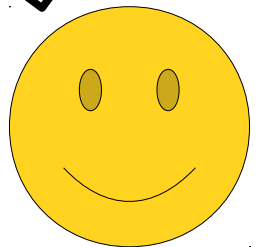
 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg p \vee (q \rightarrow \neg r)$ $p \wedge q \rightarrow \neg r$ $p \rightarrow q \rightarrow \neg r$ $\neg p \vee \neg q \vee \neg r$

...to here.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

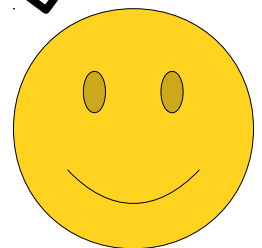
 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg p \vee (q \rightarrow \neg r)$ $p \wedge q \rightarrow \neg r$ $p \rightarrow q \rightarrow \neg r$ $\neg p \vee \neg q \vee \neg r$

And we're done!

A totally different, totally valid
negation.



$\neg\neg A$

 A $\neg(A \vee B)$

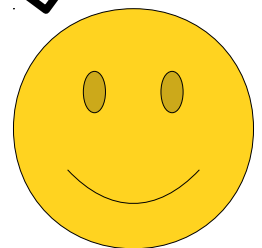
 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg p \vee (q \rightarrow \neg r)$ $p \wedge q \rightarrow \neg r$ $p \rightarrow q \rightarrow \neg r$ $\neg p \vee \neg q \vee \neg r$ $p \rightarrow (\neg q \vee \neg r)$ $(p \rightarrow \neg q) \vee \neg r$

For reference, here are all the statements you can get to using the above templates. Is your answer on that list?



$\neg\neg A$

 A $\neg(A \vee B)$

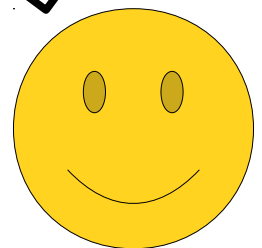
 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg p \vee (q \rightarrow \neg r)$ $p \wedge q \rightarrow \neg r$ $p \rightarrow q \rightarrow \neg r$ $\neg p \vee \neg q \vee \neg r$ $p \rightarrow (\neg q \vee \neg r)$ $(p \rightarrow \neg q) \vee \neg r$

So why did we go through this
(admittedly lengthy) diversion?



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg p \vee (q \rightarrow \neg r)$ $p \wedge q \rightarrow \neg r$ $p \rightarrow q \rightarrow \neg r$ $\neg p \vee \neg q \vee \neg r$ $p \rightarrow (\neg q \vee \neg r)$ $(p \rightarrow \neg q) \vee \neg r$

Well, any of these formulas are valid negations of the original formula.



$\neg\neg A$

 A $\neg(A \vee B)$

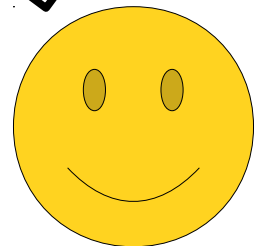
 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg p \vee (q \rightarrow \neg r)$ $p \wedge q \rightarrow \neg r$ $p \rightarrow q \rightarrow \neg r$ $\neg p \vee \neg q \vee \neg r$ $p \rightarrow (\neg q \vee \neg r)$ $(p \rightarrow \neg q) \vee \neg r$

However, as you can see, not all of them are very easy to read.



$\neg\neg A$

 A $\neg(A \vee B)$

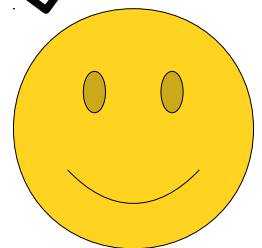
 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg p \vee (q \rightarrow \neg r)$ $p \wedge q \rightarrow \neg r$ $p \rightarrow q \rightarrow \neg r$ $\neg p \vee \neg q \vee \neg r$ $p \rightarrow (\neg q \vee \neg r)$ $(p \rightarrow \neg q) \vee \neg r$

We personally think that these two are the cleanest of the negations.



$\neg\neg A$

 A $\neg(A \vee B)$

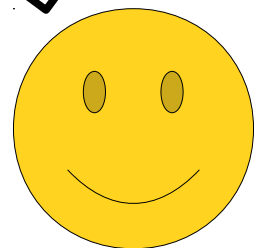
 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg p \vee (q \rightarrow \neg r)$ $p \wedge q \rightarrow \neg r$ $p \rightarrow q \rightarrow \neg r$ $\neg p \vee \neg q \vee \neg r$ $p \rightarrow (\neg q \vee \neg r)$ $(p \rightarrow \neg q) \vee \neg r$

Although we've stressed the importance of working through these negations using templates, there still is a bit of an art to it.



$\neg\neg A$

 A $\neg(A \vee B)$

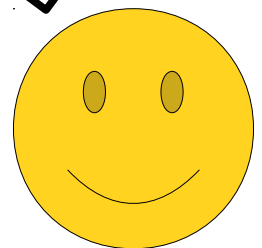
 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg p \vee (q \rightarrow \neg r)$ $p \wedge q \rightarrow \neg r$ $p \rightarrow q \rightarrow \neg r$ $\neg p \vee \neg q \vee \neg r$ $p \rightarrow (\neg q \vee \neg r)$ $(p \rightarrow \neg q) \vee \neg r$

You'll sometimes have a choice about which templates to use, and sometimes one choice will be much better than the others.



$\neg\neg A$

 A $\neg(A \vee B)$

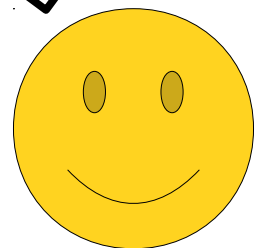
 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg p \vee (q \rightarrow \neg r)$ $p \wedge q \rightarrow \neg r$ $p \rightarrow q \rightarrow \neg r$ $\neg p \vee \neg q \vee \neg r$ $p \rightarrow (\neg q \vee \neg r)$ $(p \rightarrow \neg q) \vee \neg r$

Learning which rules to use takes some practice, but once you get the hang of it, it's really not too bad!



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg p \vee (q \rightarrow \neg r)$ $p \wedge q \rightarrow \neg r$ $p \rightarrow q \rightarrow \neg r$ $\neg p \vee \neg q \vee \neg r$ $p \rightarrow (\neg q \vee \neg r)$ $(p \rightarrow \neg q) \vee \neg r$

If you ever make an "inelegant" choice, just back up and choose something else!



$\neg\neg A$

 A $\neg(A \vee B)$

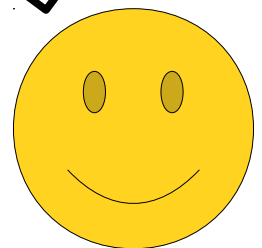
 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$

To see this in action, let's talk about how to negate biconditionals.



$\neg\neg A$

 A $\neg(A \vee B)$

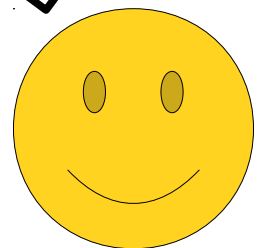
 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \leftrightarrow B)$

 $A \rightarrow \neg B$

Although we didn't talk about it in class, there are two nice rules you can use to negate a biconditional.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

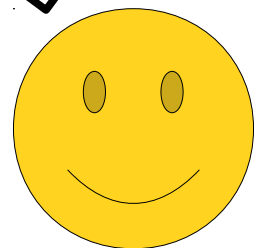
 $A \wedge \neg B$ $\neg(A \leftrightarrow B)$

 $\neg A \leftrightarrow B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$

 $A \leftrightarrow \neg B$

They're shown above. Basically, you negate one side, leave the other alone, and leave the connective unchanged.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

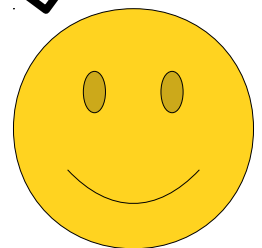
 $A \wedge \neg B$ $\neg(A \leftrightarrow B)$

 $\neg A \leftrightarrow B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$

 $A \leftrightarrow \neg B$

(As an aside, the biconditional is the only connective that doesn't change when negated. Nifty!)



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \leftrightarrow B)$

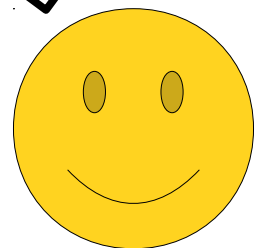
 $\neg A \leftrightarrow B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$

 $A \leftrightarrow \neg B$

$$((p \wedge q) \vee r) \leftrightarrow ((q \vee r) \rightarrow p)$$

so let's put these new rules to practice by negating the statement to the left.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

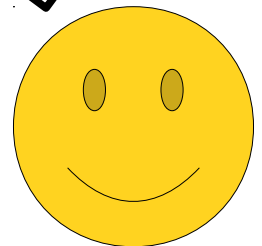
 $A \wedge \neg B$ $\neg(A \leftrightarrow B)$

 $\neg A \leftrightarrow B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$

 $A \leftrightarrow \neg B$
$$((p \wedge q) \vee r) \leftrightarrow ((q \vee r) \rightarrow p)$$

This one looks tricky! Don't worry - it's not as bad as it looks.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

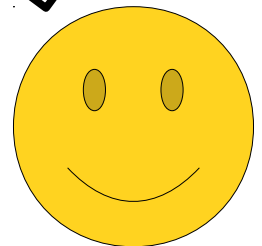
 $A \wedge \neg B$ $\neg(A \leftrightarrow B)$

 $\neg A \leftrightarrow B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$

 $A \leftrightarrow \neg B$
$$\neg(((p \wedge q) \vee r) \leftrightarrow ((q \vee r) \rightarrow p))$$

We start off the same way that we always do - parenthesize and negate.



$$\neg\neg A$$

$$A$$

$$\neg(A \vee B)$$

$$\neg A \wedge \neg B$$

$$\neg(A \wedge B)$$

$$\neg A \vee \neg B$$

$$\neg(A \wedge B)$$

$$A \rightarrow \neg B$$

$$\neg(A \rightarrow B)$$

$$A \wedge \neg B$$

$$\neg(A \leftrightarrow B)$$

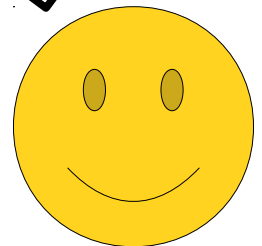
$$\neg A \leftrightarrow B$$

$$\neg(A \leftrightarrow B)$$

$$A \leftrightarrow \neg B$$

$$\neg(((p \wedge q) \vee r) \leftrightarrow ((q \vee r) \rightarrow p))$$

Now, we need to apply some template here, so we need to pick one of the two biconditional templates.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \wedge B)$

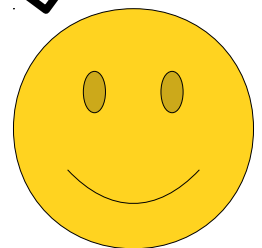
 $A \rightarrow \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \leftrightarrow B)$

 $\neg A \leftrightarrow B$ $\neg(A \leftrightarrow B)$

 $A \leftrightarrow \neg B$
$$\neg(((p \wedge q) \vee r) \leftrightarrow ((q \vee r) \rightarrow p))$$

Now, we have a choice. We can either negate the left-hand side or the right-hand side.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \wedge B)$

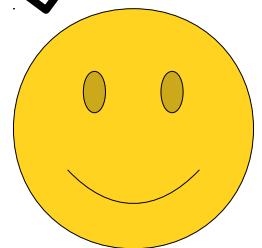
 $A \rightarrow \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \leftrightarrow B)$

 $\neg A \leftrightarrow B$ $\neg(A \leftrightarrow B)$

 $A \leftrightarrow \neg B$
$$\neg(((p \wedge q) \vee r) \leftrightarrow ((q \vee r) \rightarrow p))$$

Either way will work, but one way is a lot easier than the other.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

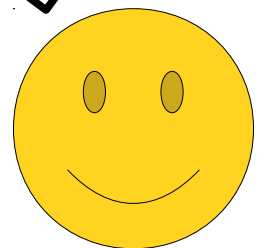
 $A \wedge \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$

 $\neg A \leftrightarrow B$ $\neg(A \leftrightarrow B)$

 $A \leftrightarrow \neg B$
$$\neg(((p \wedge q) \vee r) \leftrightarrow ((q \vee r) \rightarrow p))$$

Let's think a step ahead.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \leftrightarrow B)$

 $\neg A \leftrightarrow B$ $\neg(A \leftrightarrow B)$

 $A \leftrightarrow \neg B$ $\neg(((p \wedge q) \vee r) \leftrightarrow ((q \vee r) \rightarrow p))$

If we negate this side, we're going to have to negate the OR, and that's going to require us to then negate the AND.



$$\neg\neg A$$

$$A$$

$$\neg(A \vee B)$$

$$\neg A \wedge \neg B$$

$$\neg(A \wedge B)$$

$$\neg A \vee \neg B$$

$$\neg(A \wedge B)$$

$$A \rightarrow \neg B$$

$$\neg(A \rightarrow B)$$

$$A \wedge \neg B$$

$$\neg(A \leftrightarrow B)$$

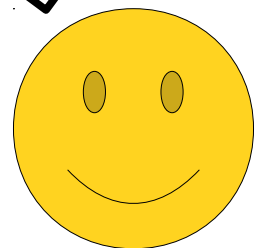
$$\neg A \leftrightarrow B$$

$$\neg(A \leftrightarrow B)$$

$$A \leftrightarrow \neg B$$

$$\neg(((p \wedge q) \vee r) \leftrightarrow ((q \vee r) \rightarrow p))$$

On the other hand, if we negate this side, we just negate the implication, which is going to leave the OR unchanged.



$$\neg\neg A$$

$$A$$

$$\neg(A \vee B)$$

$$\neg A \wedge \neg B$$

$$\neg(A \wedge B)$$

$$\neg A \vee \neg B$$

$$\neg(A \rightarrow B)$$

$$A \wedge \neg B$$

$$\neg(A \leftrightarrow B)$$

$$\neg A \leftrightarrow B$$

$$\neg(A \wedge B)$$

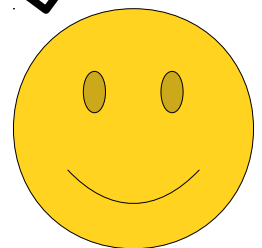
$$A \rightarrow \neg B$$

$$\neg(A \leftrightarrow B)$$

$$A \leftrightarrow \neg B$$

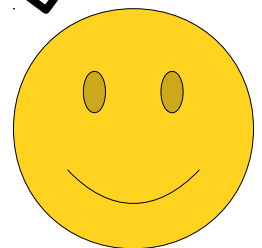
$$\neg(((p \wedge q) \vee r) \leftrightarrow ((q \vee r) \rightarrow p))$$

As a result, let's try applying this template, which negates the right-hand side.



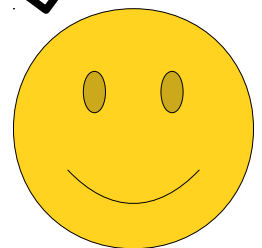
$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg(((p \wedge q) \vee r) \leftrightarrow ((q \vee r) \rightarrow p))$

Here's the before shot...



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $((p \wedge q) \vee r) \leftrightarrow \neg((q \vee r) \rightarrow p)$

... and the after.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \leftrightarrow B)$

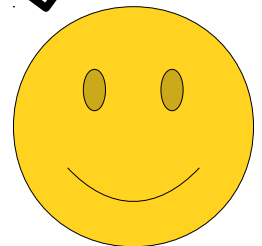
 $\neg A \leftrightarrow B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$

 $A \leftrightarrow \neg B$

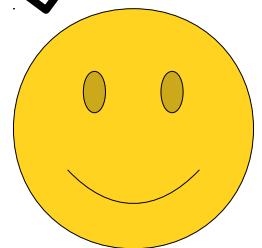
 $((p \wedge q) \vee r) \leftrightarrow \neg((q \vee r) \rightarrow p)$

So now we need to push this negation deeper.



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $((p \wedge q) \vee r) \leftrightarrow \neg((q \vee r) \rightarrow p)$

That's a relatively straightforward application of this template.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

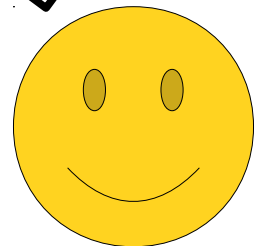
 $A \wedge \neg B$ $\neg(A \leftrightarrow B)$

 $\neg A \leftrightarrow B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$

 $A \leftrightarrow \neg B$ $((p \wedge q) \vee r) \leftrightarrow \neg((q \vee r) \rightarrow p)$

So we color-code for convenience...



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

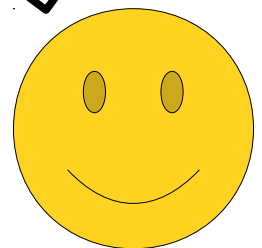
 $A \wedge \neg B$ $\neg(A \leftrightarrow B)$

 $\neg A \leftrightarrow B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$

 $A \leftrightarrow \neg B$ $((p \wedge q) \vee r) \leftrightarrow ((q \vee r) \wedge \neg p)$

...and simplify.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \leftrightarrow B)$

 $\neg A \leftrightarrow B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$

 $A \leftrightarrow \neg B$

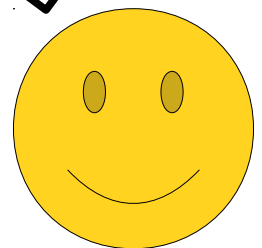
$$((p \wedge q) \vee r) \leftrightarrow ((q \vee r) \wedge \neg p)$$

And bam! We're done.



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $((p \wedge q) \vee r) \leftrightarrow ((q \vee r) \wedge \neg p)$ $((p \wedge q) \vee r) \leftrightarrow ((q \vee r) \rightarrow p)$

You can - and should - check that this formula is correct by using the truth table tool. Here's the original as a reminder.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

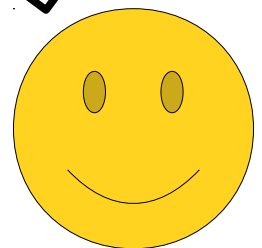
 $A \wedge \neg B$ $\neg(A \leftrightarrow B)$

 $\neg A \leftrightarrow B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$

 $A \leftrightarrow \neg B$

So wow! We've covered a lot
of ground. Let's recap.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \leftrightarrow B)$

 $\neg A \leftrightarrow B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$

 $A \leftrightarrow \neg B$

We've recently introduced new templates for negating implications and biconditionals.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \leftrightarrow B)$

 $\neg A \leftrightarrow B$ $\neg(A \leftrightarrow B)$

 $A \leftrightarrow \neg B$

You've seen that there's also a different way to negate ANDs using implications.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \leftrightarrow B)$

 $\neg A \leftrightarrow B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$

 $A \leftrightarrow \neg B$

And we touched a bit on the idea that there's an art to choosing these templates wisely.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

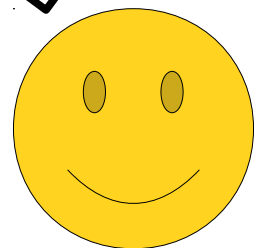
 $A \wedge \neg B$ $\neg(A \leftrightarrow B)$

 $\neg A \leftrightarrow B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$

 $A \leftrightarrow \neg B$

That's basically everything we'd like to say about negating formulas in propositional logic.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

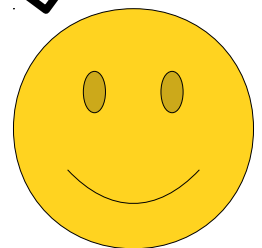
 $A \wedge \neg B$ $\neg(A \leftrightarrow B)$

 $\neg A \leftrightarrow B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$

 $A \leftrightarrow \neg B$

To wrap things up, let's talk about how to negate first-order logic formulas. Don't worry! They follow the same basic set of rules.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

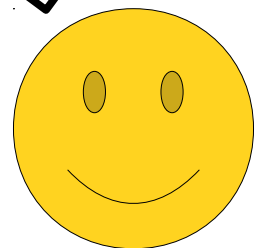
 $A \wedge \neg B$ $\neg(A \leftrightarrow B)$

 $\neg A \leftrightarrow B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$

 $A \leftrightarrow \neg B$

Fundamentally, first-order logic is quite different from propositional logic. Structurally, though, it's still propositions joined by connectives.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

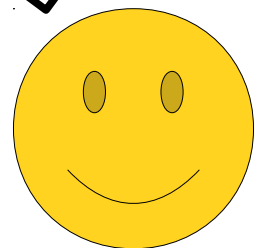
 $A \wedge \neg B$ $\neg(A \leftrightarrow B)$

 $\neg A \leftrightarrow B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$

 $A \leftrightarrow \neg B$

The only major difference is the introduction of quantifiers. Once you know how to negate those, you can negate first-order formulas with ease!



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

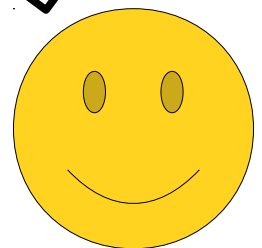
 $A \wedge \neg B$ $\neg(A \leftrightarrow B)$

 $\neg A \leftrightarrow B$ $\neg(A \wedge B)$

 $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$

 $A \leftrightarrow \neg B$

Let's see this with an example.



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$

$$\forall x. (Person(x) \rightarrow$$

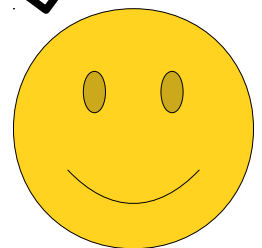
$$\exists y. (Person(y) \wedge y \neq x \wedge$$

$$Loves(x, y)$$

$$)$$

$$)$$

Here's a formula from class.
Do you remember what this says?



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$

$$\forall x. (Person(x) \rightarrow$$

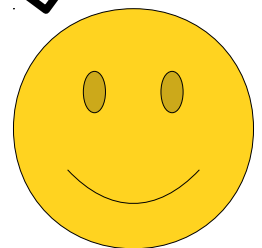
$$\exists y. (Person(y) \wedge y \neq x \wedge$$

$$Loves(x, y)$$

$$)$$

$$)$$

Let's see how to negate it.



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$

$$\neg \forall x. (Person(x) \rightarrow$$

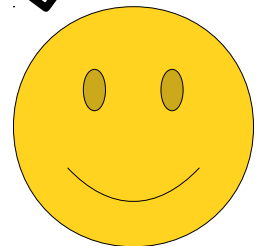
$$\exists y. (Person(y) \wedge y \neq x \wedge$$

$$Loves(x, y)$$

$$)$$

$$)$$

As before, we put a negation at the front. (Since this whole formula is controlled by the quantifier, we don't need to parenthesize it.)



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$

$$\neg \forall x. (Person(x) \rightarrow$$

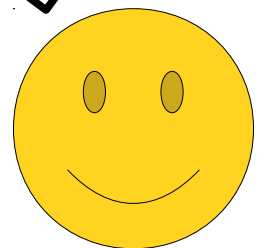
$$\exists y. (Person(y) \wedge y \neq x \wedge$$

$$Loves(x, y)$$

$$)$$

$$)$$

Now, we need some kind of template for simplifying this expression.



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$

$$\neg \forall x. (Person(x) \rightarrow$$

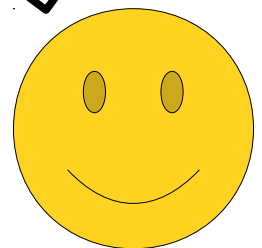
$$\exists y. (Person(y) \wedge y \neq x \wedge$$

$$Loves(x, y)$$

$$)$$

$$)$$

If you'll remember from lecture, we saw how to negate quantifiers: push the negation across the quantifier, then flip the quantifier.



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\neg\forall x. (Person(x) \rightarrow$$

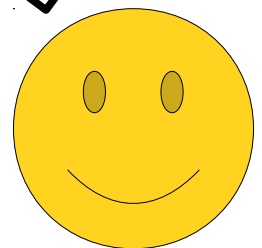
$$\exists y. (Person(y) \wedge y \neq x \wedge$$

$$Loves(x, y)$$

$$)$$

$$)$$

Those rules look like this.



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\neg\forall x. (Person(x) \rightarrow$$

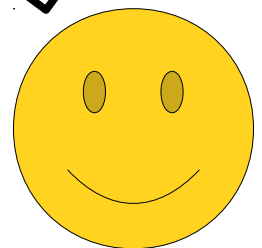
$$\exists y. (Person(y) \wedge y \neq x \wedge$$

$$Loves(x, y)$$

$$)$$

$$)$$

so let's go apply them!

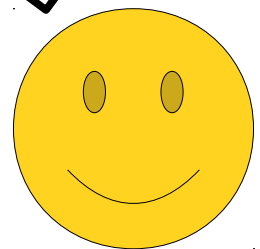


$\neg\neg A$	$\neg(A \vee B)$	$\neg(A \wedge B)$	$\neg(A \rightarrow B)$	$\neg(A \leftrightarrow B)$
<hr/>	<hr/>	<hr/>	<hr/>	<hr/>
A	$\neg A \wedge \neg B$	$\neg A \vee \neg B$	$A \wedge \neg B$	$\neg A \leftrightarrow B$
		$\neg(A \wedge B)$		$\neg(A \leftrightarrow B)$
		<hr/>		<hr/>
		$A \rightarrow \neg B$		$A \leftrightarrow \neg B$

$\neg\forall x. A$	$\neg\exists x. A$
<hr/>	<hr/>
$\exists x. \neg A$	$\forall x. \neg A$

$\neg\forall x. (Person(x) \rightarrow$
 $\exists y. (Person(y) \wedge y \neq x \wedge$
 $Loves(x, y)$
 $)$
 $)$

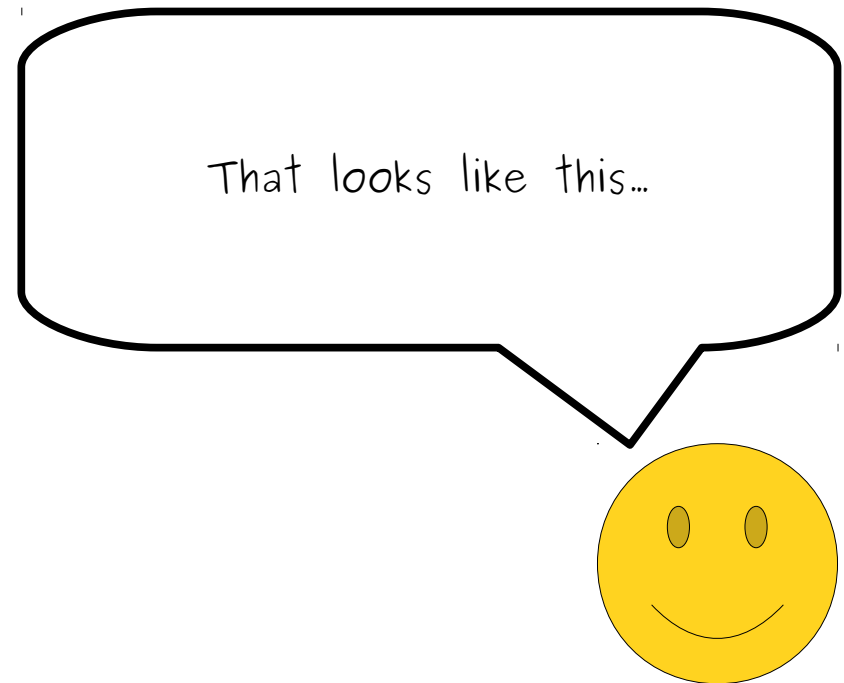
Here, we can apply this template.



$\neg\neg A$	$\neg(A \vee B)$	$\neg(A \wedge B)$	$\neg(A \rightarrow B)$	$\neg(A \leftrightarrow B)$
A	$\neg A \wedge \neg B$	$\neg A \vee \neg B$	$A \wedge \neg B$	$\neg A \leftrightarrow B$
		$\neg(A \wedge B)$		$\neg(A \leftrightarrow B)$
		$A \rightarrow \neg B$		$A \leftrightarrow \neg B$

$\neg\forall x. A$	$\neg\exists x. A$
$\exists x. \neg A$	$\forall x. \neg A$

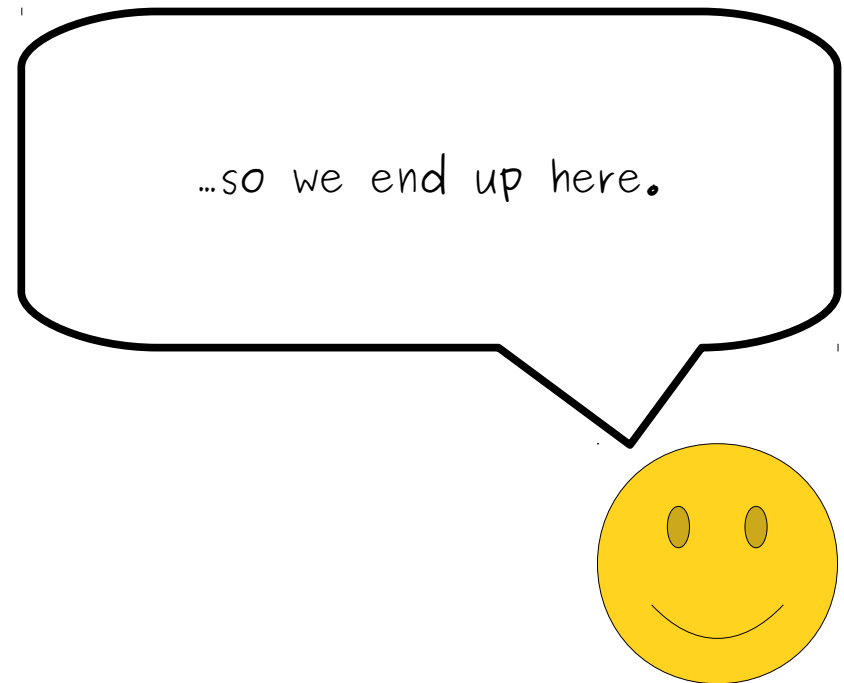
$\neg\forall x. (Person(x) \rightarrow$
 $\exists y. (Person(y) \wedge y \neq x \wedge$
 $Loves(x, y)$
 $)$
 $)$



$\neg\neg A$	$\neg(A \vee B)$	$\neg(A \wedge B)$	$\neg(A \rightarrow B)$	$\neg(A \leftrightarrow B)$
A	$\neg A \wedge \neg B$	$\neg A \vee \neg B$	$A \wedge \neg B$	$\neg A \leftrightarrow B$
		$\neg(A \wedge B)$		$\neg(A \leftrightarrow B)$
		$A \rightarrow \neg B$		$A \leftrightarrow \neg B$

$\neg\forall x. A$	$\neg\exists x. A$
$\exists x. \neg A$	$\forall x. \neg A$

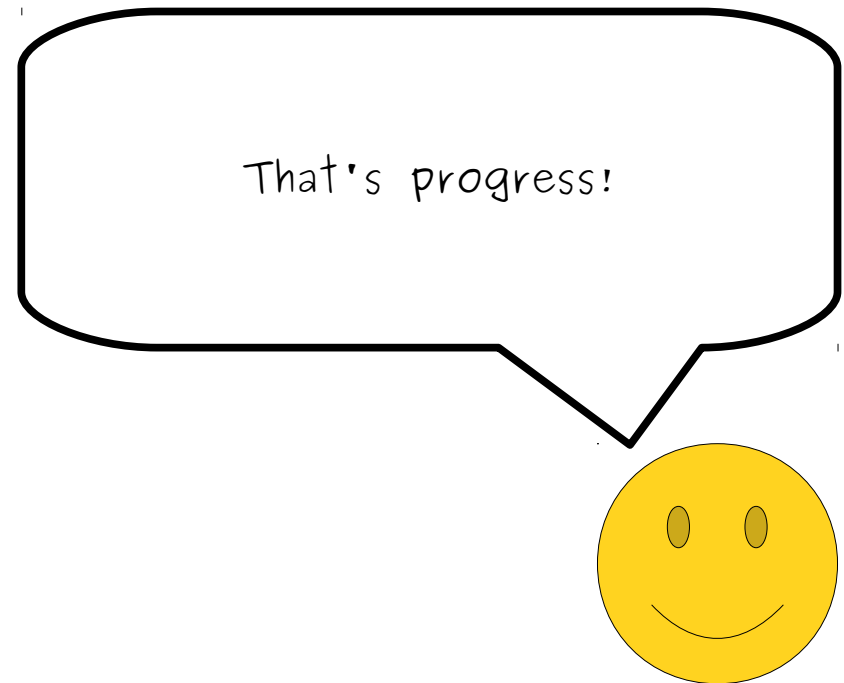
$\exists x. \neg(Person(x) \rightarrow$
 $\exists y. (Person(y) \wedge y \neq x \wedge$
 $Loves(x, y)$
 $)$
 $)$



$\neg\neg A$	$\neg(A \vee B)$	$\neg(A \wedge B)$	$\neg(A \rightarrow B)$	$\neg(A \leftrightarrow B)$
A	$\neg A \wedge \neg B$	$\neg A \vee \neg B$	$A \wedge \neg B$	$\neg A \leftrightarrow B$
		$\neg(A \wedge B)$		$\neg(A \leftrightarrow B)$
		$A \rightarrow \neg B$		$A \leftrightarrow \neg B$

$\neg\forall x. A$	$\neg\exists x. A$
$\exists x. \neg A$	$\forall x. \neg A$

$\exists x. \neg(Person(x) \rightarrow$
 $\exists y. (Person(y) \wedge y \neq x \wedge$
 $Loves(x, y)$
 $)$
 $)$



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\exists x. \neg(Person(x) \rightarrow$$

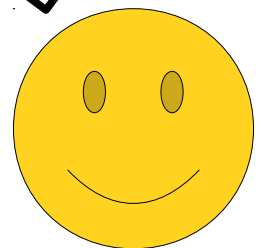
$$\exists y. (Person(y) \wedge y \neq x \wedge$$

$$Loves(x, y)$$

$$)$$

$$)$$

Now, how do we continue?



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\exists x. \neg(Person(x) \rightarrow$$

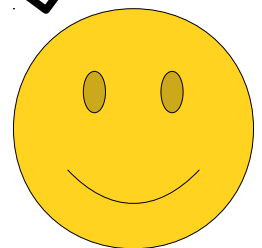
$$\exists y. (Person(y) \wedge y \neq x \wedge$$

$$Loves(x, y)$$

$$)$$

$$)$$

Well, notice that we have a negation applied to an implication.



$$\frac{\neg\neg A}{A}$$

$$\frac{\neg(A \vee B)}{\neg A \wedge \neg B}$$

$$\frac{\neg(A \wedge B)}{\neg A \vee \neg B}$$

$$\frac{\neg(A \rightarrow B)}{A \wedge \neg B}$$

$$\frac{\neg(A \leftrightarrow B)}{\neg A \leftrightarrow B}$$

$$\frac{\neg(A \wedge B)}{A \rightarrow \neg B}$$

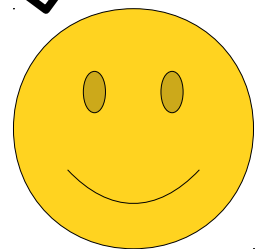
$$\frac{\neg(A \leftrightarrow B)}{A \leftrightarrow \neg B}$$

$$\frac{\neg\forall x. A}{\exists x. \neg A}$$

$$\frac{\neg\exists x. A}{\forall x. \neg A}$$

$\exists x. \neg(Person(x) \rightarrow$
 $\exists y. (Person(y) \wedge y \neq x \wedge$
 $Loves(x, y)$
 $)$
 $)$

Well, notice that we have a negation applied to an implication.



$$\frac{\neg\neg A}{A}$$

$$\frac{\neg(A \vee B)}{\neg A \wedge \neg B}$$

$$\frac{\neg(A \wedge B)}{\neg A \vee \neg B}$$

$$\frac{\neg(A \rightarrow B)}{A \wedge \neg B}$$

$$\frac{\neg(A \leftrightarrow B)}{\neg A \leftrightarrow B}$$

$$\frac{\neg(A \wedge B)}{A \rightarrow \neg B}$$

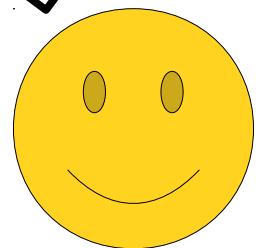
$$\frac{\neg(A \leftrightarrow B)}{A \leftrightarrow \neg B}$$

$$\frac{\neg\forall x. A}{\exists x. \neg A}$$

$$\frac{\neg\exists x. A}{\forall x. \neg A}$$

$\exists x. \neg(Person(x) \rightarrow$
 $\exists y. (Person(y) \wedge y \neq x \wedge$
 $Loves(x, y)$
 $)$
 $)$

That suggests we should use this template.



$$\frac{\neg\neg A}{A}$$

$$\frac{\neg(A \vee B)}{\neg A \wedge \neg B}$$

$$\frac{\neg(A \wedge B)}{\neg A \vee \neg B}$$

$$\frac{\neg(A \rightarrow B)}{A \wedge \neg B}$$

$$\frac{\neg(A \leftrightarrow B)}{\neg A \leftrightarrow B}$$

$$\frac{\neg(A \wedge B)}{A \rightarrow \neg B}$$

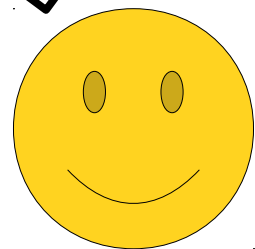
$$\frac{\neg(A \leftrightarrow B)}{A \leftrightarrow \neg B}$$

$$\frac{\neg\forall x. A}{\exists x. \neg A}$$

$$\frac{\neg\exists x. A}{\forall x. \neg A}$$

$\exists x. \neg(Person(x) \rightarrow$
 $\exists y. (Person(y) \wedge y \neq x \wedge$
 $Loves(x, y)$
 $)$
 $)$

Here's the antecedent and consequent, in case that makes things clearer.



$$\frac{\neg\neg A}{A}$$

$$\frac{\neg(A \vee B)}{\neg A \wedge \neg B}$$

$$\frac{\neg(A \wedge B)}{\neg A \vee \neg B}$$

$$\frac{\neg(A \rightarrow B)}{A \wedge \neg B}$$

$$\frac{\neg(A \leftrightarrow B)}{\neg A \leftrightarrow B}$$

$$\frac{\neg(A \wedge B)}{A \rightarrow \neg B}$$

$$\frac{\neg(A \leftrightarrow B)}{A \leftrightarrow \neg B}$$

$$\frac{\neg\forall x. A}{\exists x. \neg A}$$

$$\frac{\neg\exists x. A}{\forall x. \neg A}$$

$\exists x. \neg(Person(x) \rightarrow$
 $\exists y. (Person(y) \wedge y \neq x \wedge$
 $Loves(x, y)$
 $)$
 $)$

Applying the rule...



$$\frac{\neg\neg A}{A}$$

$$\frac{\neg(A \vee B)}{\neg A \wedge \neg B}$$

$$\frac{\neg(A \wedge B)}{\neg A \vee \neg B}$$

$$\frac{\neg(A \rightarrow B)}{A \wedge \neg B}$$

$$\frac{\neg(A \leftrightarrow B)}{\neg A \leftrightarrow B}$$

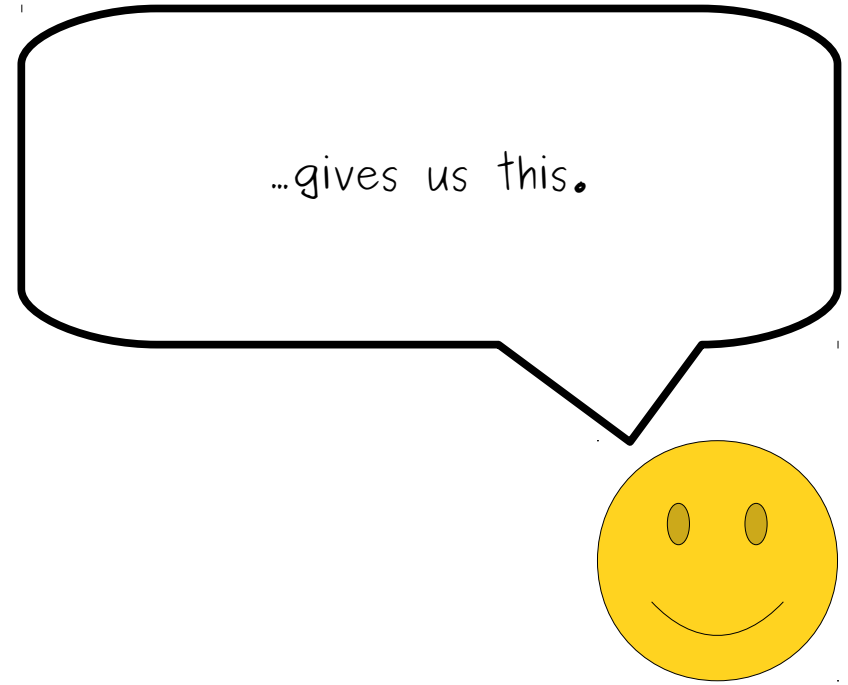
$$\frac{\neg(A \wedge B)}{A \rightarrow \neg B}$$

$$\frac{\neg(A \leftrightarrow B)}{A \leftrightarrow \neg B}$$

$$\frac{\neg\forall x. A}{\exists x. \neg A}$$

$$\frac{\neg\exists x. A}{\forall x. \neg A}$$

$\exists x. (Person(x) \wedge$
 $\neg\exists y. (Person(y) \wedge y \neq x \wedge$
 $Loves(x, y)$
 $)$
 $)$



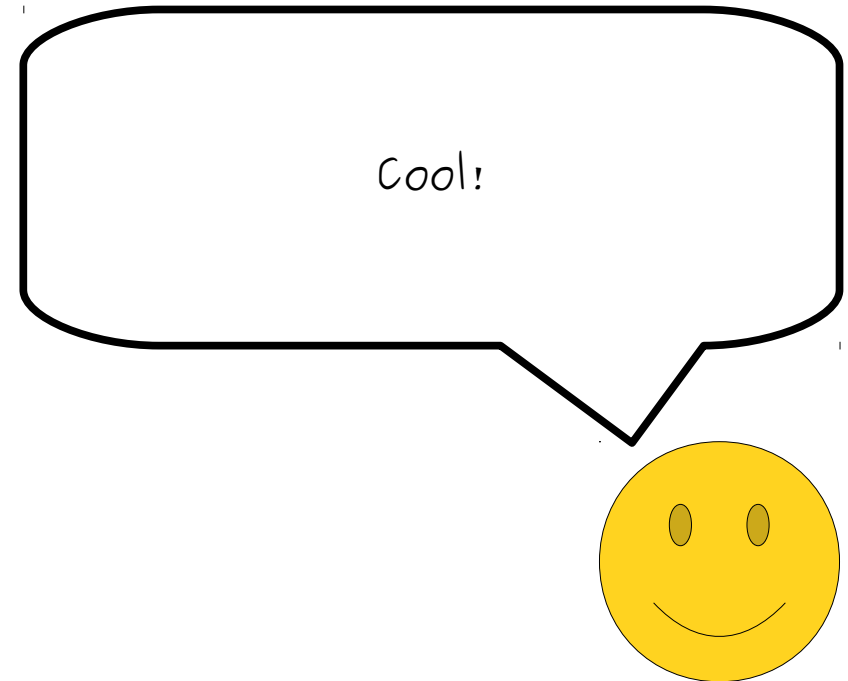
$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\exists x. (Person(x) \wedge$$

$$\neg\exists y. (Person(y) \wedge y \neq x \wedge$$

$$Loves(x, y))$$

$$)$$

$$)$$


$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\exists x. (Person(x) \wedge$$

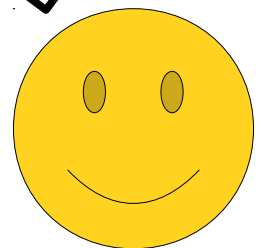
$$\neg\exists y. (Person(y) \wedge y \neq x \wedge$$

$$Loves(x, y)$$

$$)$$

$$)$$

It's just a matter of repeating
this process until we're done.



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\exists x. (Person(x) \wedge$$

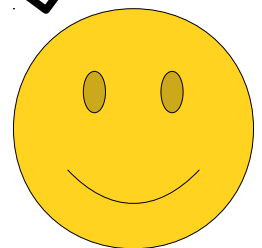
$$\neg\exists y. (Person(y) \wedge y \neq x \wedge$$

$$Loves(x, y)$$

$$)$$

$$)$$

Here we have a negated
existential statement...

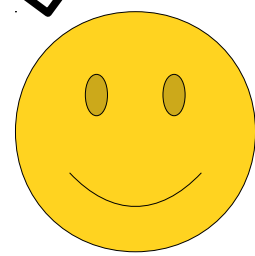


$\frac{\neg\neg A}{A}$	$\frac{\neg(A \vee B)}{\neg A \wedge \neg B}$	$\frac{\neg(A \wedge B)}{\neg A \vee \neg B}$	$\frac{\neg(A \rightarrow B)}{A \wedge \neg B}$	$\frac{\neg(A \leftrightarrow B)}{\neg A \leftrightarrow B}$
		$\frac{\neg(A \wedge B)}{A \rightarrow \neg B}$		$\frac{\neg(A \leftrightarrow B)}{A \leftrightarrow \neg B}$

$\frac{\neg\forall x. A}{\exists x. \neg A}$	$\frac{\neg\exists x. A}{\forall x. \neg A}$
--	--

$\exists x. (Person(x) \wedge$
 $\forall y. \neg(Person(y) \wedge y \neq x \wedge$
 $Loves(x, y)$
 $)$
 $)$

...which simplifies like this.



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\exists x. (Person(x) \wedge$$

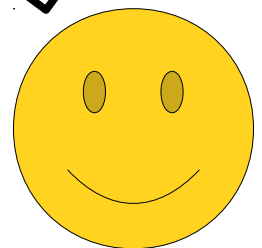
$$\forall y. \neg(Person(y) \wedge y \neq x \wedge$$

$$Loves(x, y)$$

$$)$$

$$)$$

Great! Now, we have a negated AND. It's a three-way AND, actually, but fortunately we've seen lots of ways to handle this!



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\exists x. (Person(x) \wedge$$

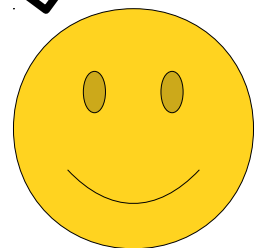
$$\forall y. \neg(Person(y) \wedge y \neq x \wedge$$

$$Loves(x, y)$$

$$)$$

$$)$$

While we can use any template we'd like here, when dealing with first-order logic, it's often useful to use this rule, which turns ANDs into implications.



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\exists x. (Person(x) \wedge$$

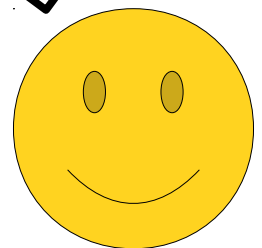
$$\forall y. \neg(Person(y) \wedge y \neq x \wedge$$

$$Loves(x, y)$$

$$)$$

$$)$$

We'll group things like this...



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\exists x. (Person(x) \wedge$$

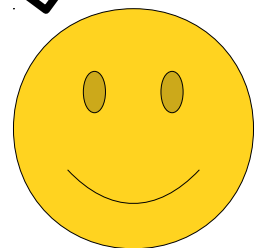
$$\forall y. (Person(y) \wedge y \neq x \rightarrow$$

$$\neg Loves(x, y))$$

$$)$$

$$)$$

... which gives us this as a result.



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\exists x. (Person(x) \wedge$$

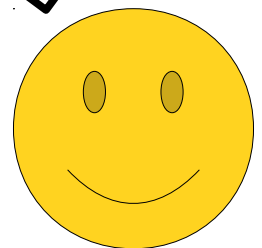
$$\forall y. (Person(y) \wedge y \neq x \rightarrow$$

$$\neg Loves(x, y))$$

$$)$$

$$)$$

And now we're done!



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\exists x. (Person(x) \wedge$$

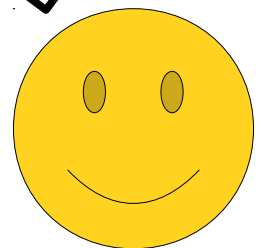
$$\forall y. (Person(y) \wedge y \neq x \rightarrow$$

$$\neg Loves(x, y))$$

$$)$$

$$)$$

So how can we check whether
this is right?



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\exists x. (Person(x) \wedge$$

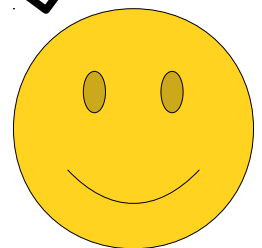
$$\forall y. (Person(y) \wedge y \neq x \rightarrow$$

$$\neg Loves(x, y))$$

$$)$$

$$)$$

When dealing with propositional logic, we could just plug our result into the truth table tool.



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\exists x. (Person(x) \wedge$$

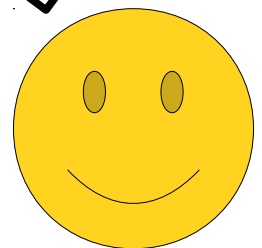
$$\forall y. (Person(y) \wedge y \neq x \rightarrow$$

$$\neg Loves(x, y))$$

$$)$$

$$)$$

Unfortunately, first-order logic doesn't have truth tables, so that particular approach isn't going to work.



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\exists x. (Person(x) \wedge$$

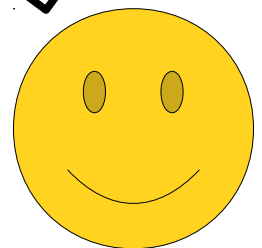
$$\forall y. (Person(y) \wedge y \neq x \rightarrow$$

$$\neg Loves(x, y))$$

$$)$$

$$)$$

It turns out, more generally, that it is objectively hard to check whether one first-order formula is the negation of another.



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\exists x. (Person(x) \wedge$$

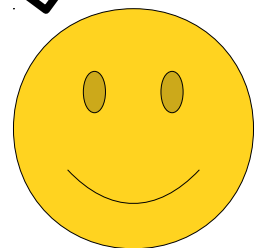
$$\forall y. (Person(y) \wedge y \neq x \rightarrow$$

$$\neg Loves(x, y))$$

$$)$$

$$)$$

You know how we've been talking about problems so hard that they can't be solved by a computer? This is one of them!



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\exists x. (Person(x) \wedge$$

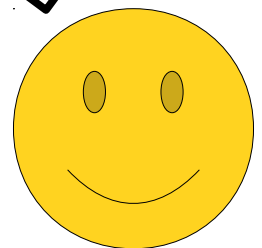
$$\forall y. (Person(y) \wedge y \neq x \rightarrow$$

$$\neg Loves(x, y))$$

$$)$$

$$)$$

We're not going to talk about that in CS103. Take Phil 152 or CS154 for details!



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\exists x. (Person(x) \wedge$$

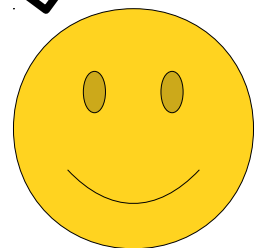
$$\forall y. (Person(y) \wedge y \neq x \rightarrow$$

$$\neg Loves(x, y))$$

$$)$$

$$)$$

So if there's no simple way to confirm that we have the right answer, how do we know that we've gotten it right?



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\exists x. (Person(x) \wedge$$

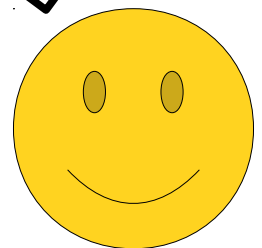
$$\forall y. (Person(y) \wedge y \neq x \rightarrow$$

$$\neg Loves(x, y))$$

$$)$$

$$)$$

At some level, we have to trust that we applied the templates correctly.



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\exists x. (Person(x) \wedge$$

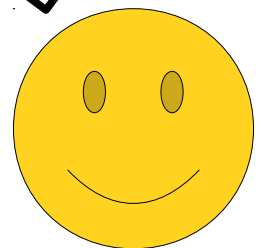
$$\forall y. (Person(y) \wedge y \neq x \rightarrow$$

$$\neg Loves(x, y))$$

$$)$$

$$)$$

However, we still can do a few quick checks to make sure we didn't mess anything up.



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\exists x. (Person(x) \wedge$$

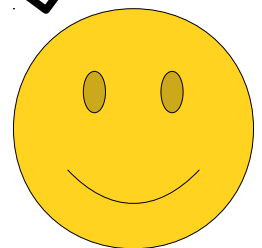
$$\forall y. (Person(y) \wedge y \neq x \rightarrow$$

$$\neg Loves(x, y))$$

$$)$$

$$)$$

For example, we know that \forall is (usually) paired with \rightarrow and that \exists is (usually) paired with \wedge .



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\exists x. (Person(x) \wedge$$

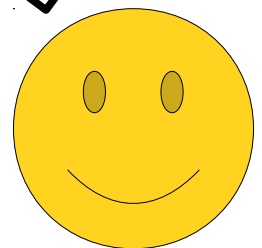
$$\forall y. (Person(y) \wedge y \neq x \rightarrow$$

$$\neg Loves(x, y))$$

$$)$$

$$)$$

A great way to check your work is to make sure that you didn't break that rule. If you broke the rule, chances are you made a mistake somewhere.



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\exists x. (Person(x) \wedge$$

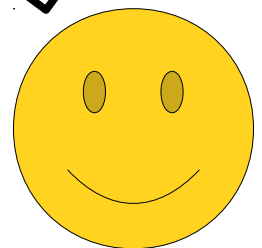
$$\forall y. (Person(y) \wedge y \neq x \rightarrow$$

$$\neg Loves(x, y))$$

$$)$$

$$)$$

Here, notice that we do indeed obey the rules.



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\exists x. (Person(x) \wedge$$

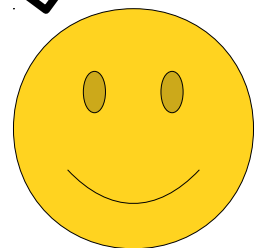
$$\forall y. (Person(y) \wedge y \neq x \rightarrow$$

$$\neg Loves(x, y))$$

$$)$$

$$)$$

This is a good sign that we didn't make any major mistakes.



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\exists x. (Person(x) \wedge$$

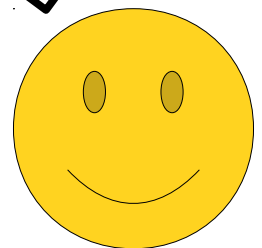
$$\forall y. (Person(y) \wedge y \neq x \rightarrow$$

$$\neg Loves(x, y))$$

$$)$$

$$)$$

To check whether we really have it, it's often useful to translate the original and negated statements into English and to check those.



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\exists x. (Person(x) \wedge$$

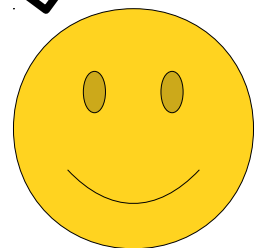
$$\forall y. (Person(y) \wedge y \neq x \rightarrow$$

$$\neg Loves(x, y))$$

$$)$$

$$)$$

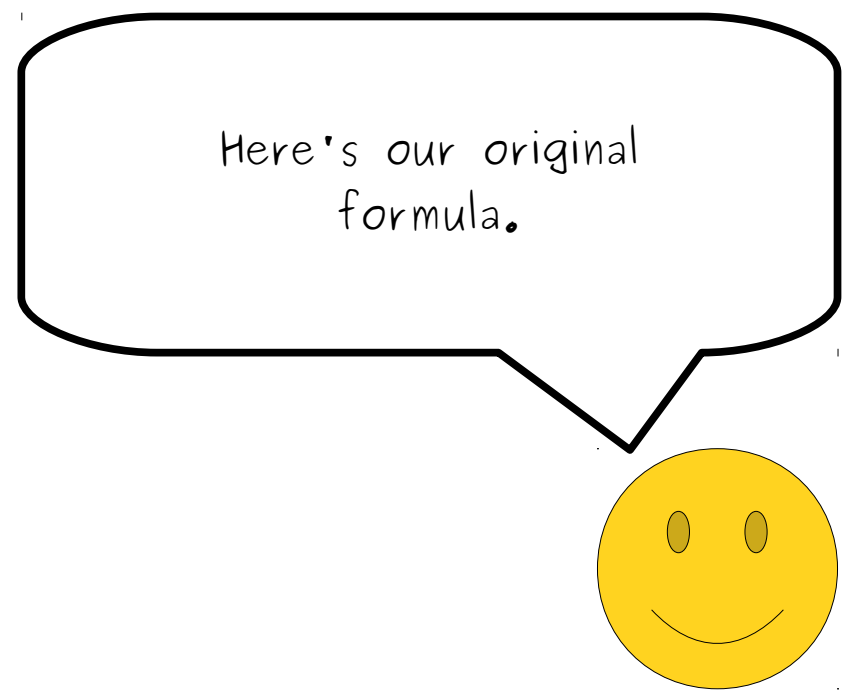
This statement says
 "there is someone that doesn't
 love anyone else."



$\neg\neg A$	$\neg(A \vee B)$	$\neg(A \wedge B)$	$\neg(A \rightarrow B)$	$\neg(A \leftrightarrow B)$
A	$\neg A \wedge \neg B$	$\neg A \vee \neg B$	$A \wedge \neg B$	$\neg A \leftrightarrow B$
		$\neg(A \wedge B)$		$\neg(A \leftrightarrow B)$
		$A \rightarrow \neg B$		$A \leftrightarrow \neg B$

$\neg\forall x. A$	$\neg\exists x. A$
$\exists x. \neg A$	$\forall x. \neg A$

$\forall x. (Person(x) \rightarrow$
 $\exists y. (Person(y) \wedge y \neq x \wedge$
 $Loves(x, y)$
 $)$
 $)$



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\forall x. (Person(x) \rightarrow$$

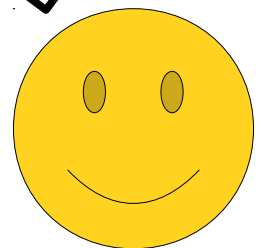
$$\exists y. (Person(y) \wedge y \neq x \wedge$$

$$Loves(x, y)$$

$$)$$

$$)$$

This says "everyone loves someone else."



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\exists x. (Person(x) \wedge$$

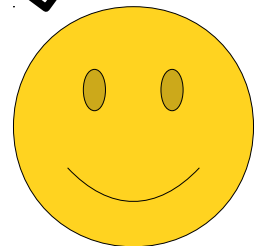
$$\forall y. (Person(y) \wedge y \neq x \rightarrow$$

$$\neg Loves(x, y))$$

$$)$$

$$)$$

(Let's reset back to the negated formula.)



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\exists x. (Person(x) \wedge$$

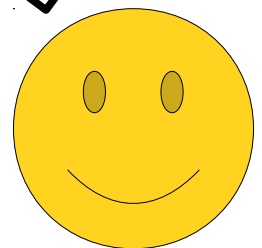
$$\forall y. (Person(y) \wedge y \neq x \rightarrow$$

$$\neg Loves(x, y))$$

$$)$$

$$)$$

The two translations are indeed negations of one another.



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\exists x. (Person(x) \wedge$$

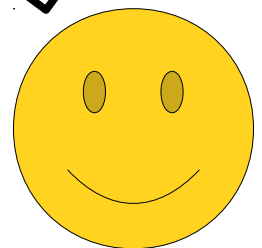
$$\forall y. (Person(y) \wedge y \neq x \rightarrow$$

$$\neg Loves(x, y))$$

$$)$$

$$)$$

If everyone loves someone else,
there's no way that there could
be a person who doesn't love
anyone else.



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\exists x. (Person(x) \wedge$$

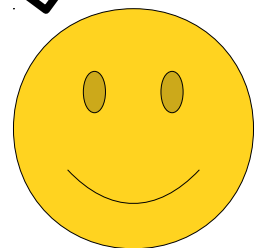
$$\forall y. (Person(y) \wedge y \neq x \rightarrow$$

$$\neg Loves(x, y))$$

$$)$$

$$)$$

Similarly, if there's someone who doesn't love anyone else, then it's definitely not the case that everyone loves someone else.



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\exists x. (Person(x) \wedge$$

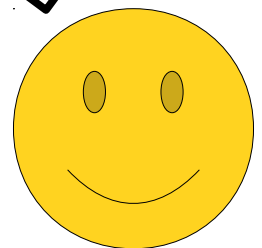
$$\forall y. (Person(y) \wedge y \neq x \rightarrow$$

$$\neg Loves(x, y))$$

$$)$$

$$)$$

So overall we should be pretty confident that we've properly negated our formula.



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\exists x. (Person(x) \wedge$$

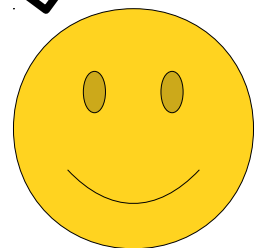
$$\forall y. (Person(y) \wedge y \neq x \rightarrow$$

$$\neg Loves(x, y))$$

$$)$$

$$)$$

It passes our basic structural tests, and also intuitively syncs up with what the formulas mean.



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\exists x. (Person(x) \wedge$$

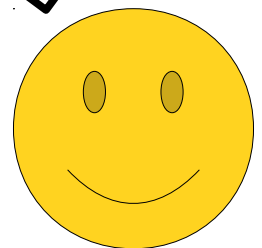
$$\forall y. (Person(y) \wedge y \neq x \rightarrow$$

$$\neg Loves(x, y))$$

$$)$$

$$)$$

Before we move on, I wanted to touch on one question: why did we use the "AND-to-implies" template to negate the ANDs here?



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\exists x. (Person(x) \wedge$$

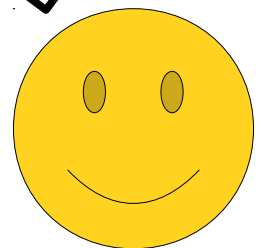
$$\forall y. (Person(y) \wedge y \neq x \rightarrow$$

$$\neg Loves(x, y))$$

$$)$$

$$)$$

To answer that, let's roll back the clock to the point where we made that choice.



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\exists x. (Person(x) \wedge$$

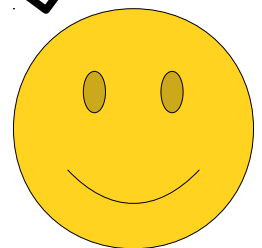
$$\forall y. \neg(Person(y) \wedge y \neq x \wedge$$

$$Loves(x, y)$$

$$)$$

$$)$$

Here we are!



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\exists x. (Person(x) \wedge$$

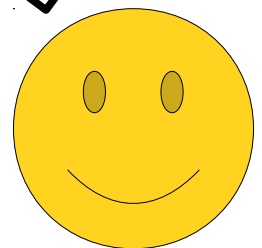
$$\forall y. \neg(Person(y) \wedge y \neq x \wedge$$

$$Loves(x, y)$$

$$)$$

$$)$$

Now, we saw a number of ways that we can negate a three-way AND. (It's almost like we did that example intentionally to set this discussion up!)



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\exists x. (Person(x) \wedge$$

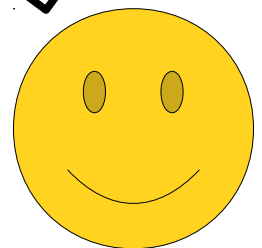
$$\forall y. \neg(Person(y) \wedge y \neq x \wedge$$

$$Loves(x, y)$$

$$)$$

$$)$$

Let's apply de Morgan's laws here, using the rule that $\neg(A \wedge B \wedge C) \equiv \neg A \vee \neg B \vee \neg C$



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\exists x. (Person(x) \wedge$$

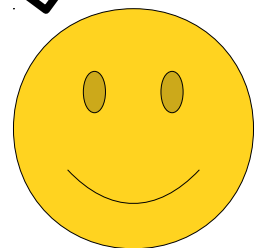
$$\forall y. (\neg Person(y) \vee \neg y \neq x \vee$$

$$\neg Loves(x, y)$$

$$)$$

$$)$$

That gives us this result.



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\exists x. (Person(x) \wedge$$

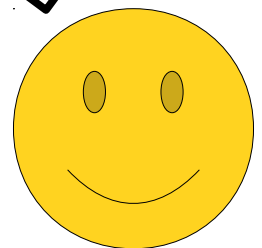
$$\forall y. (\neg Person(y) \vee \neg y \neq x \vee$$

$$\neg Loves(x, y)$$

$$)$$

$$)$$

So this result is correct - we know this because we applied valid rules the whole way through...



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\exists x. (Person(x) \wedge$$

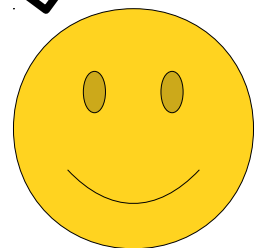
$$\forall y. (\neg Person(y) \vee \neg y \neq x \vee$$

$$\neg Loves(x, y)$$

$$)$$

$$)$$

...but it's a lot harder to read this statement. That quantifier reads "any y either isn't a person, or is equal to x, or x doesn't love y."



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\exists x. (Person(x) \wedge$$

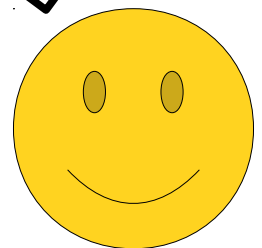
$$\forall y. (\neg Person(y) \vee \neg y \neq x \vee$$

$$\neg Loves(x, y)$$

$$)$$

$$)$$

That's quite a mouthful.



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\exists x. (Person(x) \wedge$$

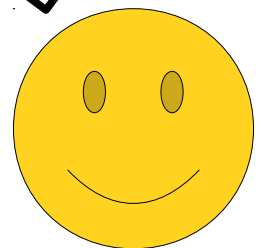
$$\forall y. (\neg Person(y) \vee \neg y \neq x \vee$$

$$\neg Loves(x, y)$$

$$)$$

$$)$$

Plus, this breaks the usual pattern of \forall getting paired with \rightarrow , which makes it harder to check.



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\exists x. (Person(x) \wedge$$

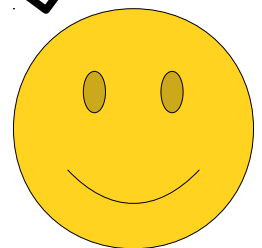
$$\forall y. (\neg Person(y) \vee \neg y \neq x \vee$$

$$\neg Loves(x, y)$$

$$)$$

$$)$$

One of the reasons why the "AND-to-implies" rule is so useful is that it preserves which connectives pair with which quantifiers.



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$$\exists x. (Person(x) \wedge$$

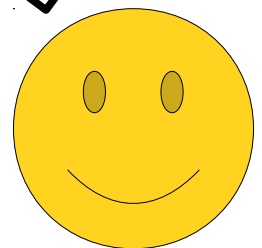
$$\forall y. (\neg Person(y) \vee \neg y \neq x \vee$$

$$\neg Loves(x, y)$$

$$)$$

$$)$$

This is why we recommend that you not use de Morgan's laws with AND and instead go with the other route. It's just a lot easier!



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \leftrightarrow B)$

 $\neg A \leftrightarrow B$ $\neg(A \wedge B)$

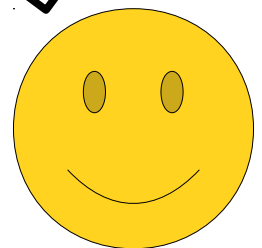
 $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$

 $A \leftrightarrow \neg B$ $\neg\forall x. A$

 $\exists x. \neg A$ $\neg\exists x. A$

 $\forall x. \neg A$

Having said that, let's go and do another, more complex example.



$$\neg\neg A$$

$$A$$

$$\neg(A \vee B)$$

$$\neg A \wedge \neg B$$

$$\neg(A \wedge B)$$

$$\neg A \vee \neg B$$

$$\neg(A \wedge B)$$

$$A \rightarrow \neg B$$

$$\neg(A \rightarrow B)$$

$$A \wedge \neg B$$

$$\neg(A \leftrightarrow B)$$

$$\neg A \leftrightarrow B$$

$$\neg(A \leftrightarrow B)$$

$$A \leftrightarrow \neg B$$

$$\neg\forall x. A$$

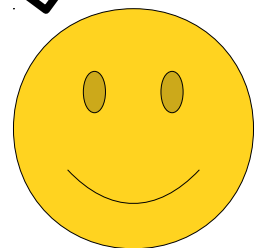
$$\exists x. \neg A$$

$$\neg\exists x. A$$

$$\forall x. \neg A$$

$\forall S. \forall T. (Set(S) \wedge Set(T) \rightarrow$
 $(S = T \leftrightarrow \forall x. (x \in S \leftrightarrow x \in T))$
)

Here's a formula from set theory. Before we go on, can you translate this into English?



$$\neg\neg A$$

$$A$$

$$\neg(A \vee B)$$

$$\neg A \wedge \neg B$$

$$\neg(A \wedge B)$$

$$\neg A \vee \neg B$$

$$\neg(A \wedge B)$$

$$A \rightarrow \neg B$$

$$\neg(A \rightarrow B)$$

$$A \wedge \neg B$$

$$\neg(A \leftrightarrow B)$$

$$\neg A \leftrightarrow B$$

$$\neg(A \leftrightarrow B)$$

$$A \leftrightarrow \neg B$$

$$\neg\forall x. A$$

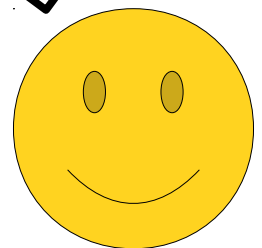
$$\exists x. \neg A$$

$$\neg\exists x. A$$

$$\forall x. \neg A$$

$\forall S. \forall T. (Set(S) \wedge Set(T) \rightarrow$
 $(S = T \leftrightarrow \forall x. (x \in S \leftrightarrow x \in T))$
)

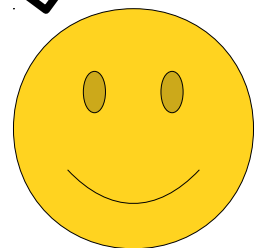
This says "any two sets are equal if and only if they have the same elements." If you're not sure why, dive into this and see if you can convince yourself!



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$\forall S. \forall T. (Set(S) \wedge Set(T) \rightarrow$
 $(S = T \leftrightarrow \forall x. (x \in S \leftrightarrow x \in T))$
 $)$

Let's see how to negate this.



$$\neg\neg A$$

$$A$$

$$\neg(A \vee B)$$

$$\neg A \wedge \neg B$$

$$\neg(A \wedge B)$$

$$\neg A \vee \neg B$$

$$\neg(A \wedge B)$$

$$A \rightarrow \neg B$$

$$\neg(A \rightarrow B)$$

$$A \wedge \neg B$$

$$\neg(A \leftrightarrow B)$$

$$\neg A \leftrightarrow B$$

$$\neg(A \leftrightarrow B)$$

$$A \leftrightarrow \neg B$$

$$\neg\forall x. A$$

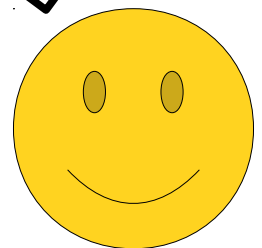
$$\exists x. \neg A$$

$$\neg\exists x. A$$

$$\forall x. \neg A$$

$\forall S. \forall T. (Set(S) \wedge Set(T) \rightarrow$
 $(S = T \leftrightarrow \forall x. (x \in S \leftrightarrow x \in T))$
 $)$

Actually, let's not! Let's begin
by having you take a stab at it.
Try negating this one!



$$\neg\neg A$$

$$A$$

$$\neg(A \vee B)$$

$$\neg A \wedge \neg B$$

$$\neg(A \wedge B)$$

$$\neg A \vee \neg B$$

$$\neg(A \wedge B)$$

$$A \rightarrow \neg B$$

$$\neg(A \rightarrow B)$$

$$A \wedge \neg B$$

$$\neg(A \leftrightarrow B)$$

$$\neg A \leftrightarrow B$$

$$\neg(A \leftrightarrow B)$$

$$A \leftrightarrow \neg B$$

$$\neg\forall x. A$$

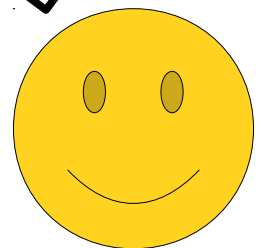
$$\exists x. \neg A$$

$$\neg\exists x. A$$

$$\forall x. \neg A$$

$\forall S. \forall T. (Set(S) \wedge Set(T) \rightarrow$
 $(S = T \leftrightarrow \forall x. (x \in S \leftrightarrow x \in T))$
 $)$

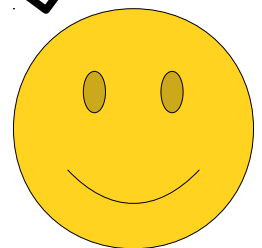
Seriously, go try to negate it.
Don't keep reading until you
do.



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$\forall S. \forall T. (Set(S) \wedge Set(T) \rightarrow$
 $(S = T \leftrightarrow \forall x. (x \in S \leftrightarrow x \in T))$
 $)$

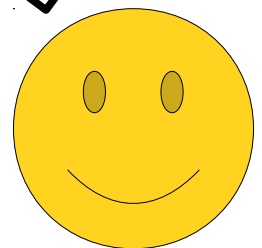
So you negated it? Like, really?
 Because once you see us do it
 you'll never get to experience the
 thrill of discovering it yourself.



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$\forall S. \forall T. (Set(S) \wedge Set(T) \rightarrow$
 $(S = T \leftrightarrow \forall x. (x \in S \leftrightarrow x \in T))$
 $)$

Okay! So let's do this!



$$\neg\neg A$$

$$A$$

$$\neg(A \vee B)$$

$$\neg A \wedge \neg B$$

$$\neg(A \wedge B)$$

$$\neg A \vee \neg B$$

$$\neg(A \wedge B)$$

$$A \rightarrow \neg B$$

$$\neg(A \rightarrow B)$$

$$A \wedge \neg B$$

$$\neg(A \leftrightarrow B)$$

$$\neg A \leftrightarrow B$$

$$\neg(A \leftrightarrow B)$$

$$A \leftrightarrow \neg B$$

$$\neg\forall x. A$$

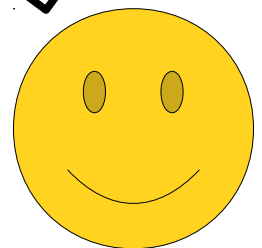
$$\exists x. \neg A$$

$$\neg\exists x. A$$

$$\forall x. \neg A$$

$$\neg\forall S. \forall T. (Set(S) \wedge Set(T) \rightarrow (S = T \leftrightarrow \forall x. (x \in S \leftrightarrow x \in T)))$$

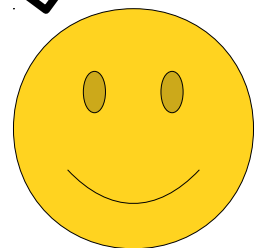
Let's begin by putting a negation at the front. Again, since the whole statement is quantified, we can pass on the parentheses.



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$\neg\forall S. \forall T. (Set(S) \wedge Set(T) \rightarrow$
 $(S = T \leftrightarrow \forall x. (x \in S \leftrightarrow x \in T))$
 $)$

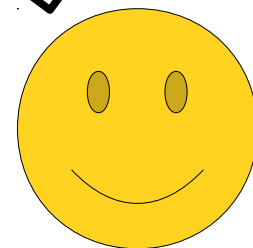
We now have something matching this template, so let's go apply it!



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$\neg\forall S. \forall T. (Set(S) \wedge Set(T) \rightarrow$
 $(S = T \leftrightarrow \forall x. (x \in S \leftrightarrow x \in T))$
 $)$

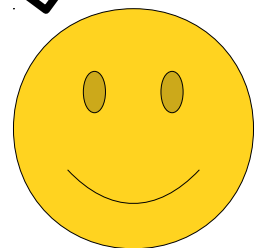
Before...



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$\exists S. \neg\forall T. (Set(S) \wedge Set(T) \rightarrow$
 $(S = T \leftrightarrow \forall x. (x \in S \leftrightarrow x \in T))$
 $)$

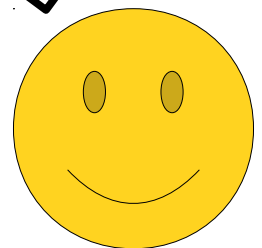
... and after!



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$\exists S. \neg\forall T. (Set(S) \wedge Set(T) \rightarrow$
 $(S = T \leftrightarrow \forall x. (x \in S \leftrightarrow x \in T))$
 $)$

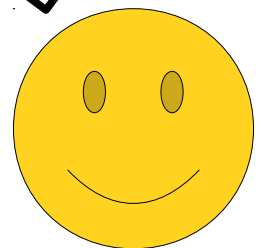
We can now apply this template
a second time to this next
quantifier.



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$\exists S. \neg\forall T. (Set(S) \wedge Set(T) \rightarrow$
 $(S = T \leftrightarrow \forall x. (x \in S \leftrightarrow x \in T))$
 $)$

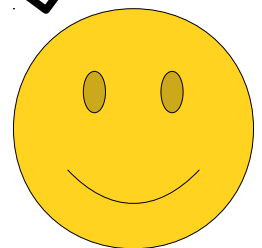
Before...



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$\exists S. \exists T. \neg(\text{Set}(S) \wedge \text{Set}(T) \rightarrow$
 $(S = T \leftrightarrow \forall x. (x \in S \leftrightarrow x \in T))$
 $)$

... and after!



$$\frac{\neg\neg A}{A}$$

$$\frac{\neg(A \vee B)}{\neg A \wedge \neg B}$$

$$\frac{\neg(A \wedge B)}{\neg A \vee \neg B}$$

$$\frac{\neg(A \rightarrow B)}{A \wedge \neg B}$$

$$\frac{\neg(A \leftrightarrow B)}{\neg A \leftrightarrow B}$$

$$\frac{\neg(A \wedge B)}{A \rightarrow \neg B}$$

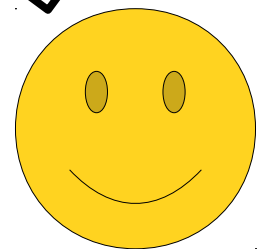
$$\frac{\neg(A \leftrightarrow B)}{A \leftrightarrow \neg B}$$

$$\frac{\neg\forall x. A}{\exists x. \neg A}$$

$$\frac{\neg\exists x. A}{\forall x. \neg A}$$

$\exists S. \exists T. \neg(\text{Set}(S) \wedge \text{Set}(T) \rightarrow$
 $(S = T \leftrightarrow \forall x. (x \in S \leftrightarrow x \in T)))$
 $)$

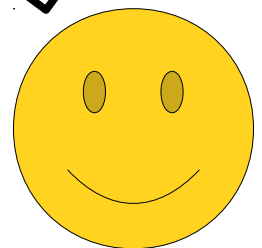
We now have a negation lined up against an implication, so let's go and apply this template.



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$\exists S. \exists T. \neg(\text{Set}(S) \wedge \text{Set}(T) \rightarrow$
 $(S = T \leftrightarrow \forall x. (x \in S \leftrightarrow x \in T))$
 $)$

Before...



$$\frac{\neg\neg A}{A}$$

$$\frac{\neg(A \vee B)}{\neg A \wedge \neg B}$$

$$\frac{\neg(A \wedge B)}{\neg A \vee \neg B}$$

$$\frac{\neg(A \rightarrow B)}{A \wedge \neg B}$$

$$\frac{\neg(A \leftrightarrow B)}{\neg A \leftrightarrow B}$$

$$\frac{\neg(A \wedge B)}{A \rightarrow \neg B}$$

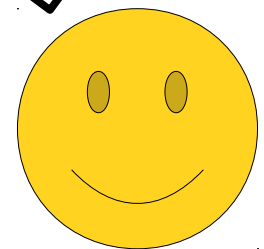
$$\frac{\neg(A \leftrightarrow B)}{A \leftrightarrow \neg B}$$

$$\frac{\neg\forall x. A}{\exists x. \neg A}$$

$$\frac{\neg\exists x. A}{\forall x. \neg A}$$

$\exists S. \exists T. (Set(S) \wedge Set(T) \wedge$
 $\neg(S = T \leftrightarrow \forall x. (x \in S \leftrightarrow x \in T))$
 $)$

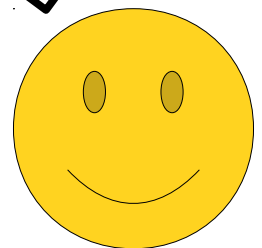
... and after!



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$\exists S. \exists T. (Set(S) \wedge Set(T) \wedge$
 $\neg(S = T \leftrightarrow \forall x. (x \in S \leftrightarrow x \in T))$
 $)$

Now our negation is sitting in front of a biconditional.



$$\frac{\neg\neg A}{A}$$

$$\frac{\neg(A \vee B)}{\neg A \wedge \neg B}$$

$$\frac{\neg(A \wedge B)}{\neg A \vee \neg B}$$

$$\frac{\neg(A \rightarrow B)}{A \wedge \neg B}$$

$$\frac{\neg(A \leftrightarrow B)}{\neg A \leftrightarrow B}$$
$$\frac{\neg(A \leftrightarrow B)}{A \leftrightarrow \neg B}$$

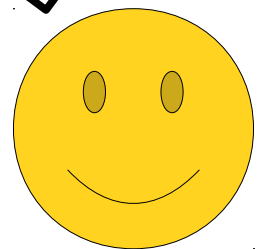
$$\frac{\neg(A \wedge B)}{A \rightarrow \neg B}$$

$$\frac{\neg\forall x. A}{\exists x. \neg A}$$

$$\frac{\neg\exists x. A}{\forall x. \neg A}$$

$\exists S. \exists T. (Set(S) \wedge Set(T) \wedge$
 $\neg(S = T \leftrightarrow \forall x. (x \in S \leftrightarrow x \in T))$
 $)$

That means we have two choices of which template to apply.



$$\frac{\neg\neg A}{A}$$

$$\frac{\neg(A \vee B)}{\neg A \wedge \neg B}$$

$$\frac{\neg(A \wedge B)}{\neg A \vee \neg B}$$

$$\frac{\neg(A \rightarrow B)}{A \wedge \neg B}$$

$$\frac{\neg(A \leftrightarrow B)}{\neg A \leftrightarrow B}$$

$$\frac{\neg(A \leftrightarrow B)}{A \leftrightarrow \neg B}$$

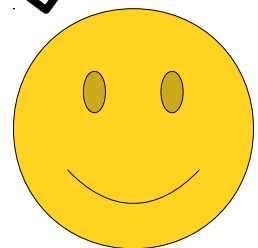
$$\frac{\neg(A \wedge B)}{A \rightarrow \neg B}$$

$$\frac{\neg\forall x. A}{\exists x. \neg A}$$

$$\frac{\neg\exists x. A}{\forall x. \neg A}$$

$\exists S. \exists T. (Set(S) \wedge Set(T) \wedge$
 $\neg(S = T \leftrightarrow \forall x. (x \in S \leftrightarrow x \in T))$
 $)$

But hey! Didn't we do some practice with that one earlier?



$$\frac{\neg\neg A}{A}$$

$$\frac{\neg(A \vee B)}{\neg A \wedge \neg B}$$

$$\frac{\neg(A \wedge B)}{\neg A \vee \neg B}$$

$$\frac{\neg(A \rightarrow B)}{A \wedge \neg B}$$

$$\frac{\neg(A \leftrightarrow B)}{\neg A \leftrightarrow B}$$
$$\frac{\neg(A \leftrightarrow B)}{A \leftrightarrow \neg B}$$

$$\frac{\neg(A \wedge B)}{A \rightarrow \neg B}$$

$$\frac{\neg\forall x. A}{\exists x. \neg A}$$

$$\frac{\neg\exists x. A}{\forall x. \neg A}$$

$\exists S. \exists T. (Set(S) \wedge Set(T) \wedge \neg(S = T \leftrightarrow \forall x. (x \in S \leftrightarrow x \in T)))$

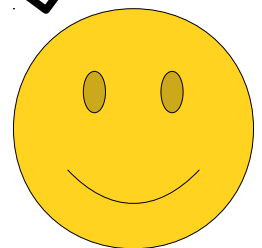
One of the lessons we took from that was that it's often useful to look at which half would be easier to negate.



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$\exists S. \exists T. (Set(S) \wedge Set(T) \wedge$
 $\neg(S = T \leftrightarrow \forall x. (x \in S \leftrightarrow x \in T))$
 $)$

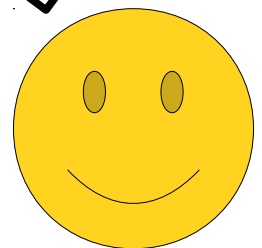
This half would be pretty easy to negate - we just change the = to a \neq .



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$\exists S. \exists T. (Set(S) \wedge Set(T) \wedge$
 $\neg(S = T \leftrightarrow \forall x. (x \in S \leftrightarrow x \in T))$
 $)$

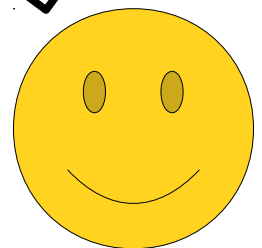
This half would be a lot harder to negate - we'd have to push the negation across the quantifier, then handle another biconditional.



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$\exists S. \exists T. (Set(S) \wedge Set(T) \wedge$
 $\neg(S = T \leftrightarrow \forall x. (x \in S \leftrightarrow x \in T))$
 $)$

Based on that, it makes a lot more sense to negate the first half of this biconditional, not the second.



$$\frac{\neg\neg A}{A}$$

$$\frac{\neg(A \vee B)}{\neg A \wedge \neg B}$$

$$\frac{\neg(A \wedge B)}{\neg A \vee \neg B}$$

$$\frac{\neg(A \rightarrow B)}{A \wedge \neg B}$$

$$\frac{\neg(A \leftrightarrow B)}{\neg A \leftrightarrow B}$$

$$\frac{\neg(A \wedge B)}{A \rightarrow \neg B}$$

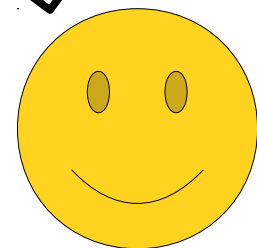
$$\frac{\neg(A \leftrightarrow B)}{A \leftrightarrow \neg B}$$

$$\frac{\neg\forall x. A}{\exists x. \neg A}$$

$$\frac{\neg\exists x. A}{\forall x. \neg A}$$

$\exists S. \exists T. (Set(S) \wedge Set(T) \wedge$
 $\neg(S = T \leftrightarrow \forall x. (x \in S \leftrightarrow x \in T))$
 $)$

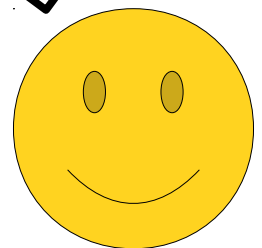
so we'll apply this template...



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$

$\exists S. \exists T. (Set(S) \wedge Set(T) \wedge$
 $(S \neq T \leftrightarrow \forall x. (x \in S \leftrightarrow x \in T))$
 $)$

...giving us this result.



$$\frac{\neg\neg A}{A}$$

$$\frac{\neg(A \vee B)}{\neg A \wedge \neg B}$$

$$\frac{\neg(A \wedge B)}{\neg A \vee \neg B}$$

$$\frac{\neg(A \rightarrow B)}{A \wedge \neg B}$$

$$\frac{\neg(A \leftrightarrow B)}{\neg A \leftrightarrow B}$$

$$\frac{\neg(A \wedge B)}{A \rightarrow \neg B}$$

$$\frac{\neg(A \leftrightarrow B)}{A \leftrightarrow \neg B}$$

$$\frac{\neg\forall x. A}{\exists x. \neg A}$$

$$\frac{\neg\exists x. A}{\forall x. \neg A}$$

$\exists S. \exists T. (Set(S) \wedge Set(T) \wedge$
 $(S \neq T \leftrightarrow \forall x. (x \in S \leftrightarrow x \in T))$
 $)$

And at this point, we're done!
Does this match what you had?



$$\frac{\neg\neg A}{A}$$

$$\frac{\neg(A \vee B)}{\neg A \wedge \neg B}$$

$$\frac{\neg(A \wedge B)}{\neg A \vee \neg B}$$

$$\frac{\neg(A \rightarrow B)}{A \wedge \neg B}$$

$$\frac{\neg(A \leftrightarrow B)}{\neg A \leftrightarrow B}$$

$$\frac{\neg(A \wedge B)}{A \rightarrow \neg B}$$

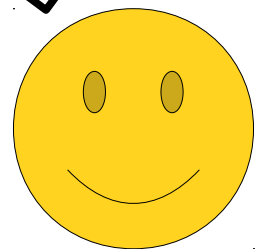
$$\frac{\neg(A \leftrightarrow B)}{A \leftrightarrow \neg B}$$

$$\frac{\neg\forall x. A}{\exists x. \neg A}$$

$$\frac{\neg\exists x. A}{\forall x. \neg A}$$

$\exists S. \exists T. (Set(S) \wedge Set(T) \wedge$
 $(S \neq T \leftrightarrow \forall x. (x \in S \leftrightarrow x \in T))$
 $)$

You might have negated the second half of the biconditional rather than the first.



$$\frac{\neg\neg A}{A}$$

$$\frac{\neg(A \vee B)}{\neg A \wedge \neg B}$$

$$\frac{\neg(A \wedge B)}{\neg A \vee \neg B}$$

$$\frac{\neg(A \rightarrow B)}{A \wedge \neg B}$$

$$\frac{\neg(A \leftrightarrow B)}{\neg A \leftrightarrow B}$$

$$\frac{\neg(A \wedge B)}{A \rightarrow \neg B}$$

$$\frac{\neg(A \leftrightarrow B)}{A \leftrightarrow \neg B}$$

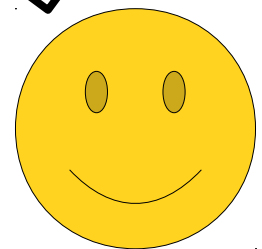
$$\frac{\neg\forall x. A}{\exists x. \neg A}$$

$$\frac{\neg\exists x. A}{\forall x. \neg A}$$

$\exists S. \exists T. (Set(S) \wedge Set(T) \wedge$
 $(S \neq T \leftrightarrow \forall x. (x \in S \leftrightarrow x \in T))$
 $)$

$\exists S. \exists T. (Set(S) \wedge Set(T) \wedge$
 $(S = T \leftrightarrow \exists x. (x \in S \leftrightarrow x \notin T))$
 $)$

If you did, you'd get something like this instead.



$$\frac{\neg\neg A}{A}$$

$$\frac{\neg(A \vee B)}{\neg A \wedge \neg B}$$

$$\frac{\neg(A \wedge B)}{\neg A \vee \neg B}$$

$$\frac{\neg(A \rightarrow B)}{A \wedge \neg B}$$

$$\frac{\neg(A \leftrightarrow B)}{\neg A \leftrightarrow B}$$

$$\frac{\neg(A \wedge B)}{A \rightarrow \neg B}$$

$$\frac{\neg(A \leftrightarrow B)}{A \leftrightarrow \neg B}$$

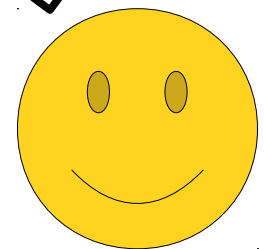
$$\frac{\neg\forall x. A}{\exists x. \neg A}$$

$$\frac{\neg\exists x. A}{\forall x. \neg A}$$

$\exists S. \exists T. (Set(S) \wedge Set(T) \wedge$
 $(S \neq T \leftrightarrow \forall x. (x \in S \leftrightarrow x \in T))$
 $)$

$\exists S. \exists T. (Set(S) \wedge Set(T) \wedge$
 $(S = T \leftrightarrow \exists x. (x \in S \leftrightarrow x \notin T))$
 $)$

If you did this, no worries!
 You're still correct. We just
 think this is a harder route to
 go down.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \leftrightarrow B)$

 $\neg A \leftrightarrow B$ $\neg(A \wedge B)$

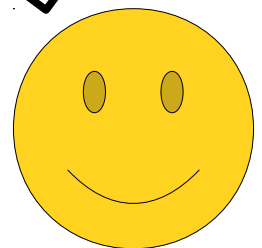
 $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$

 $A \leftrightarrow \neg B$ $\neg\forall x. A$

 $\exists x. \neg A$ $\neg\exists x. A$

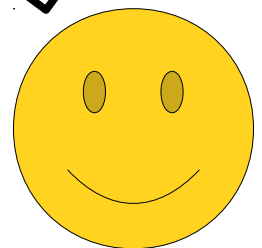
 $\forall x. \neg A$

Let's do one final example
before moving on to our
last topic.



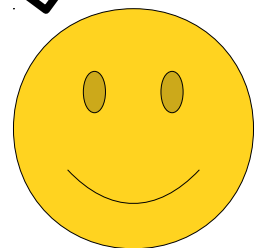
$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$ $(\forall x. \textit{Happy}(x)) \rightarrow (\exists y. \textit{Happy}(y))$

How might you negate this formula?



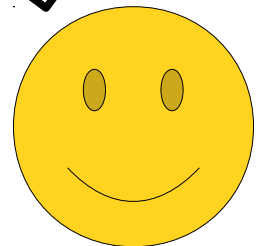
$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$ $(\forall x. \text{Happy}(x)) \rightarrow (\exists y. \text{Happy}(y))$

As before, take a few minutes to work through this one on your own time. See what you come up with!



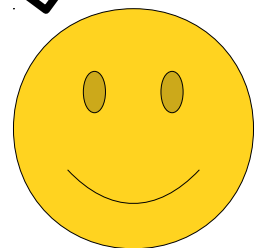
$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$ $(\forall x. \text{Happy}(x)) \rightarrow (\exists y. \text{Happy}(y))$

You know the drill. This slide is here to make sure that you actually did that work and didn't just skip by it.



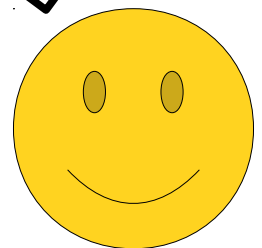
$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$ $(\forall x. \text{Happy}(x)) \rightarrow (\exists y. \text{Happy}(y))$

Okay! so let's go negate this.



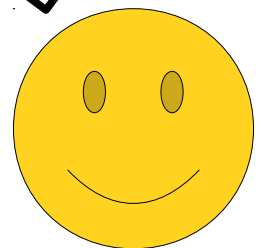
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This formula is quite different from the previous two in a subtle way.



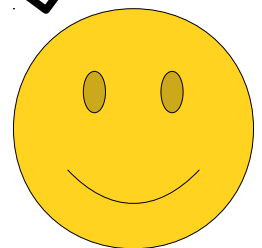
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The two previous formulas we looked at were formulas that were quantified at the top level.



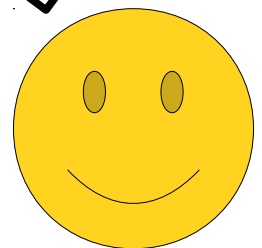
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By that, we mean that those formulas were of the form
 $\forall x. A$ or $\exists x. B$.



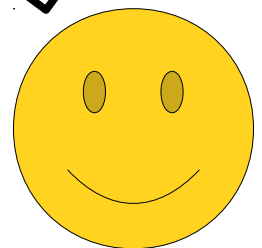
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This formula, on the other hand, is an implication where both the antecedent and consequent use quantifiers. However, the whole formula is not quantified.



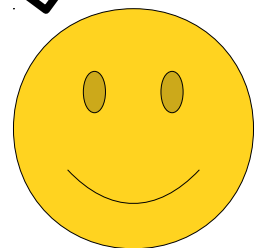
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This means that, when we start off, we'll fully parenthesize the expression and negate it, rather than just tacking a negation up-front.



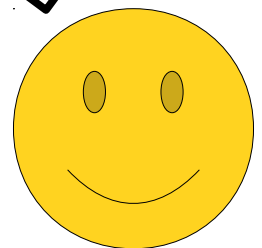
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Here's what this now looks like.



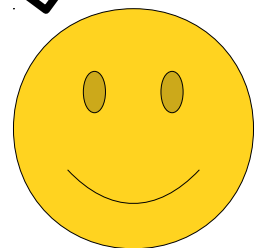
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Notice that we now have a negation applied to an implication.



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$ $\neg((\forall x. \text{Happy}(x)) \rightarrow (\exists y. \text{Happy}(y)))$

That means that we'll use this template, just as we normally do.



$$\neg\neg A$$

$$A$$

$$\neg(A \vee B)$$

$$\neg A \wedge \neg B$$

$$\neg(A \wedge B)$$

$$\neg A \vee \neg B$$

$$\neg(A \rightarrow B)$$

$$A \wedge \neg B$$

$$\neg(A \leftrightarrow B)$$

$$\neg A \leftrightarrow B$$

$$\neg(A \wedge B)$$

$$A \rightarrow \neg B$$

$$\neg(A \leftrightarrow B)$$

$$A \leftrightarrow \neg B$$

$$\neg\forall x. A$$

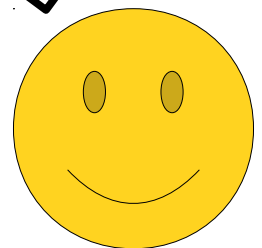
$$\exists x. \neg A$$

$$\neg\exists x. A$$

$$\forall x. \neg A$$

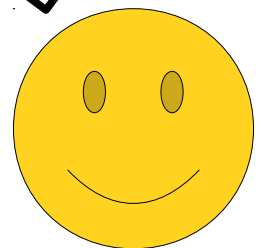
$$\neg((\forall x. \textit{Happy}(x)) \rightarrow (\exists y. \textit{Happy}(y)))$$

Here are the antecedent and consequent - each of which is a bit complicated. If we apply the template...



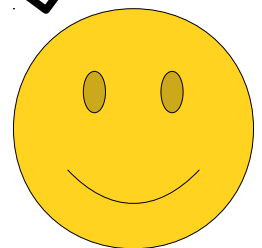
$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$ $(\forall x. \text{Happy}(x)) \wedge \neg(\exists y. \text{Happy}(y))$

... we end up here.



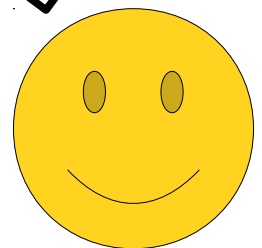
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All that's left to do is push
the negation across the
quantifier.



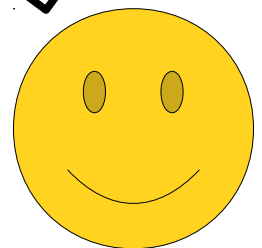
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That means we apply this rule...



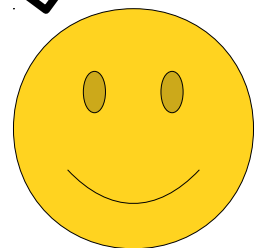
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...like this!



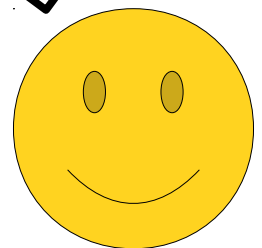
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And now we're done. Nifty!



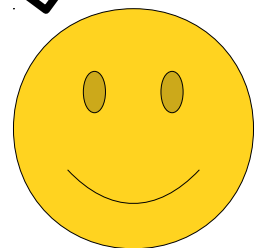
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We included this example so that you'd see that, when dealing with first-order logic, you still have to follow all the regular rules for negating statements.



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$ $(\forall x. \text{Happy}(x)) \wedge (\forall y. \neg\text{Happy}(y))$

You have to be careful to make sure that you don't negate everything, and that you figure out what the top-level structure is.



$\neg\neg A$

 A $\neg(A \vee B)$

 $\neg A \wedge \neg B$ $\neg(A \wedge B)$

 $\neg A \vee \neg B$ $\neg(A \rightarrow B)$

 $A \wedge \neg B$ $\neg(A \leftrightarrow B)$

 $\neg A \leftrightarrow B$ $\neg(A \wedge B)$

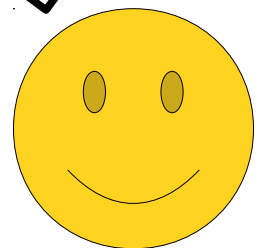
 $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$

 $A \leftrightarrow \neg B$ $\neg\forall x. A$

 $\exists x. \neg A$ $\neg\exists x. A$

 $\forall x. \neg A$

So we're almost done! We're going to do one last example.



$$\neg\neg A$$

$$A$$

$$\neg(A \vee B)$$

$$\neg A \wedge \neg B$$

$$\neg(A \wedge B)$$

$$\neg A \vee \neg B$$

$$\neg(A \rightarrow B)$$

$$A \wedge \neg B$$

$$\neg(A \leftrightarrow B)$$

$$\neg A \leftrightarrow B$$

$$\neg(A \wedge B)$$

$$A \rightarrow \neg B$$

$$\neg(A \leftrightarrow B)$$

$$A \leftrightarrow \neg B$$

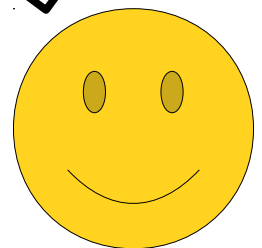
$$\neg\forall x. A$$

$$\exists x. \neg A$$

$$\neg\exists x. A$$

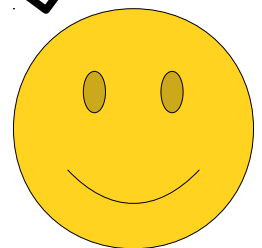
$$\forall x. \neg A$$

As you saw in lecture, sometimes we allow for quantifiers over sets.



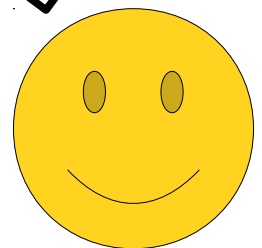
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This means that you might see formulas like this one.



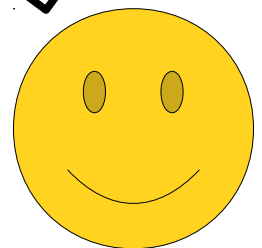
$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$ $\forall x \in S. \exists y \in S. x \neq y$

How do you negate formulas like these?



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$ $\forall x \in S. \exists y \in S. x \neq y$

The good news is that the rules for negating these "set quantifiers" are basically the same as the rules for negating the regular quantifiers.



$$\neg\neg A$$

$$A$$

$$\neg(A \vee B)$$

$$\neg A \wedge \neg B$$

$$\neg(A \wedge B)$$

$$\neg A \vee \neg B$$

$$\neg(A \rightarrow B)$$

$$A \wedge \neg B$$

$$\neg(A \leftrightarrow B)$$

$$\neg A \leftrightarrow B$$

$$\neg(A \wedge B)$$

$$A \rightarrow \neg B$$

$$\neg(A \leftrightarrow B)$$

$$A \leftrightarrow \neg B$$

$$\neg\forall x. A$$

$$\exists x. \neg A$$

$$\neg\exists x. A$$

$$\forall x. \neg A$$

$$\neg\forall x \in S. A$$

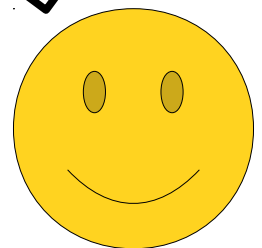
$$\exists x \in S. \neg A$$

$$\neg\exists x \in S. A$$

$$\forall x \in S. \neg A$$

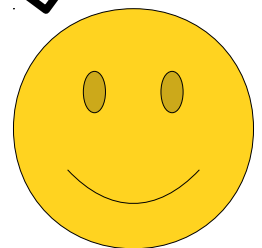
$$\forall x \in S. \exists y \in S. x \neq y$$

I've included them here. You just push the negation across the quantifier and flip which quantifier you're using.



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$ $\neg\forall x \in S. A$ $\exists x \in S. \neg A$ $\neg\exists x \in S. A$ $\forall x \in S. \neg A$ $\forall x \in S. \exists y \in S. x \neq y$

Based on that, let's finish this one off and go home. ^_^



$$\neg\neg A$$

$$A$$

$$\neg(A \vee B)$$

$$\neg A \wedge \neg B$$

$$\neg(A \wedge B)$$

$$\neg A \vee \neg B$$

$$\neg(A \rightarrow B)$$

$$A \wedge \neg B$$

$$\neg(A \leftrightarrow B)$$

$$\neg A \leftrightarrow B$$

$$\neg(A \wedge B)$$

$$A \rightarrow \neg B$$

$$\neg(A \leftrightarrow B)$$

$$A \leftrightarrow \neg B$$

$$\neg\forall x. A$$

$$\exists x. \neg A$$

$$\neg\exists x. A$$

$$\forall x. \neg A$$

$$\neg\forall x \in S. A$$

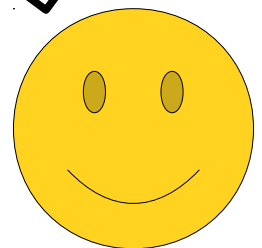
$$\exists x \in S. \neg A$$

$$\neg\exists x \in S. A$$

$$\forall x \in S. \neg A$$

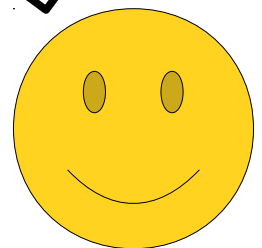
$$\neg\forall x \in S. \exists y \in S. x \neq y$$

We begin - as we are wont to do - by putting a negation in front.



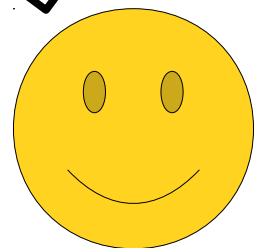
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We match this template,
so let's go apply it!



$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$ $\neg\forall x \in S. A$ $\exists x \in S. \neg A$ $\neg\exists x \in S. A$ $\forall x \in S. \neg A$ $\exists x \in S. \neg\exists y \in S. x \neq y$

Tada!



$$\neg\neg A$$

$$A$$

$$\neg(A \vee B)$$

$$\neg A \wedge \neg B$$

$$\neg(A \wedge B)$$

$$\neg A \vee \neg B$$

$$\neg(A \rightarrow B)$$

$$A \wedge \neg B$$

$$\neg(A \leftrightarrow B)$$

$$\neg A \leftrightarrow B$$

$$\neg(A \wedge B)$$

$$A \rightarrow \neg B$$

$$\neg(A \leftrightarrow B)$$

$$A \leftrightarrow \neg B$$

$$\neg\forall x. A$$

$$\exists x. \neg A$$

$$\neg\exists x. A$$

$$\forall x. \neg A$$

$$\neg\forall x \in S. A$$

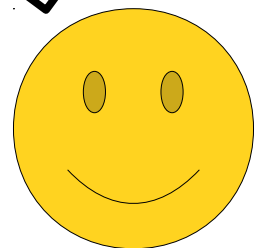
$$\exists x \in S. \neg A$$

$$\neg\exists x \in S. A$$

$$\forall x \in S. \neg A$$

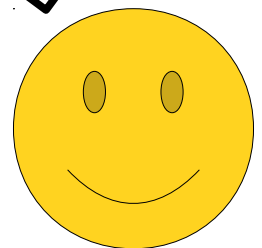
$$\exists x \in S. \neg\exists y \in S. x \neq y$$

Now we match this one...



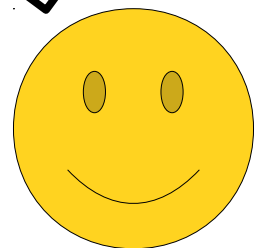
$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$ $\neg\forall x \in S. A$ $\exists x \in S. \neg A$ $\neg\exists x \in S. A$ $\forall x \in S. \neg A$ $\exists x \in S. \forall y \in S. x = y$

And we're done!



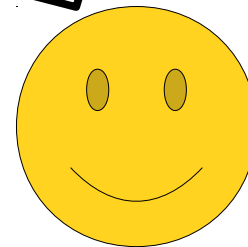
$\neg\neg A$ A $\neg(A \vee B)$ $\neg A \wedge \neg B$ $\neg(A \wedge B)$ $\neg A \vee \neg B$ $\neg(A \rightarrow B)$ $A \wedge \neg B$ $\neg(A \leftrightarrow B)$ $\neg A \leftrightarrow B$ $\neg(A \wedge B)$ $A \rightarrow \neg B$ $\neg(A \leftrightarrow B)$ $A \leftrightarrow \neg B$ $\neg\forall x. A$ $\exists x. \neg A$ $\neg\exists x. A$ $\forall x. \neg A$ $\neg\forall x \in S. A$ $\exists x \in S. \neg A$ $\neg\exists x \in S. A$ $\forall x \in S. \neg A$

So there you have it. We have a collection of templates that we can apply to negate formulas.



Hope this helps!

Please feel free to ask
questions if you have them.



Did you find this useful? If so, let us know! We can go and make more guides like these.

